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Braids in Low-Dimensional Hamiltonian Dynamics

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Contents

What is weird? [. . .] I want to argue that the weird is a particular kind of perturbation. It involves a sensation of wrongness: a weird entity or object is so strange that it makes us feel that it should not exist, or at least that it should not exist here. Yet if the entity or object is here, then the categories that we have up until now used to make sense of the world cannot be valid. The weird thing is not wrong after all: it is our conceptions that must be inadequate.

Mark Fisher, "The Weird and the Eerie"

Acknowledgements

As it turns out, writing acknowledgements is a substantially different experience from writing the rest of the thesis. The other chapters are relatively straightforward: there is something I see, I say it, and it is sorted. I guess the focal point of that kind of effort is the way things are written and justified, but overall I can claim that a non negligible proportion of what I think lies now there, on the page. Acknowledgements are another kind of beast entirely: I know that no amount of effort will ever make justice to the shards of life shared with all of you; moreover, your sheer number makes it awfully difficult to write these lines in finite time. I beg thus for your kind mercy, my readers. Let us close the braid together.

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Because the work occupying most spaces of this document is somehow in between low-dimensional topology and Hamiltonian dynamics[2], I have had the luck to cross paths with several topologists (in a broad sense) working in low dimensions, between Providence and Budapest. These people made my journeys even more special: Eric Stenhede, Fabio Capovilla-Searle, Hannah Turner, JiaJun Yan, Juan González-Meneses, Lei Chen, Marc Kegel, Marco Marengon, María Cumplido Cabello, Marithania Silvero Casanova, Michele Capovilla-Searle, Miriam Kuzbary, Orsola Capovilla-Searle, Paula Truöl, Peter Feller, Rima Chatterjee, Willi Kepplinger. Edmund and Alanna, after our first meeting in the US, you have also contributed to make my life in Paris more enjoyable

[1]He was missing from the first draft and decided, rightfully, to whine about it.
[2]No surprise here hopefully.

and interesting.

Our reading group between Aachen and Paris has been extraordinarily enriching: , I learnt a lot, beyond Persistence Modules and their far-reaching applications in Symplectic Topology. Bernhard, Jessica, Leo, Renata, Sam. I have a lot of memories with you (a surprising amount of which related to food. . .), thank you for everything and good luck, I wish you all a bright future. Tú aún no estabas en Aquisgrán, David, pero creo que lo habrías disfrutado también. Sin embargo, gracias por tu tiempo, tu paciencia y tu soporte. Marcelo and Matthias, our elder participants: thank you for your time and interest.

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An unwelcome consequence of travelling so far and wide is that, at any given moment, most of my friends are not with me[3].

The story of my life abroad started with an Erasmus at Warwick University, where I met plenty of wonderful people. Bernhard, Charles, Clothilde, David, Jakob, María, Paul, Simon: all of you marked my life some way or another, and of this I am grateful.

London friends now: Geo and David of course, Melissa and Will. Let us not forget about Lawrence, I hope he sees this and will be happy he was thanked[4]

[3]Likewise, they may complain that I am not with them. I can't know if they do though.

[4]I cannot fail to point out how he broke lasagne to make them fit in a tray. This counts as a name and shame, Lawrence, do not do it again or I will know.

Les agradezco por su hospitalidad en Zurich, Alfonso y Sonia, y por su aceite de oliva, mucho y muy sabroso. Justo ahora estoy escribiendo sentado en el sofá, aún no he partido pero ya tengo muchas ganas de volver a verlos.

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[5] E se no, parliamoci di persona così che possa ricordarvele, preferibilmente con una birra in mano.

Even though the ends of my braid are now welded together, one can always add more strands…

English Introduction

In the present dissertation we study how we can extract information about Hamiltonian diffeomorphisms of surfaces using braids obtained from Hamiltonian trajectories. We are going to approach this problem from two very different perspectives: a global point of view, in which we use braids to find estimates of Hofer distance between diffeomorphisms in a given class, and a local one, in which we focus on a specific Hamiltonian diffeomorphism. The tools we use are a new flavour of Lagrangian Floer homology, recently developed by Cristofaro-Gardiner, Humilière, Mak, Seyfaddini and Smith in [21] for the former, and for the latter generating functions for Hamiltonian diffeomorphisms (with its relations to Hamiltonian Floer homology). The results contained here will come from [51], from a joint work with Ibrahim Trifa [52] and from a yet unpublished work by the author.

0.1 Overview: setting and methods

The braid group of a surface is the fundamental group of its configuration space. Let $\Sigma_{g,p}$ be a surface with genus $g \geq 0$ and $p \geq 0$ boundary components different from the sphere. Consider the natural action of the permutation group on n letters $\mathfrak{S}_n$ on the n cartesian power of $\Sigma_{g,p}$ deprived of the fat diagonal

$$\tilde{\Delta} = \{ (x_1, \dots, x_n) | \exists i \neq j, x_i = x_j \}$$

The quotient is called n-th configuration space, and we may define the braid group $\mathcal{B}_{n,g,p}$ by

$$\mathcal{B}_{n,g,p} := \pi_1 \left((\Sigma_{g,p}^n \setminus \tilde{\Delta}) / \mathfrak{S}_n \right)$$

When $g = 0, p = 1$, i.e. in the case of the disc, we recover the usual braid group, described by the classical presentation

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i : |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} : 1 \leqslant i \leqslant n-2 \end{array} \right\rangle$$

The linking number of a braid, also called “exponent”, is the group homomorphism

$$\mathcal{B}_{n,0,1} \to \mathbb{Z}$$

which sends the generators to 1.

In this work we study Hamiltonian diffeomorphisms using the braids formed by their orbits. A Hamiltonian diffeomorphism is the flow at time 1 of a Hamiltonian vector field, which in turn is a gradient vector field of a function with respect to a symplectic form. Ultimately, the linking numbers carry information about Hamiltonian diffeomorphism because of a variational principle describing its fixed points. This principle states that critical points of the action functional, defined on the loop space of a symplectic manifold, are precisely closed Hamiltonian orbits. A Morse-like theory may be defined for the action functional: this is, philosophically, how Floer theory is defined. The generators of the Floer homology groups are then closed Hamiltonian loops, and the differential is defined counting perturbed pseudoholomorphic cylinders with asymptotics at Hamiltonian loops. It turns out that pseudoholomorphic curves (in our case, the graphs of Floer curves) in 4-manifolds intersect positively, and this fact admits a braid theoretical interpretation: the linking numbers of braids formed by Hamiltonian orbits increase along the Floer differential.

In the direction of research described in the paragraph "A linking filtration" below, we discretise the variational problem to finite dimensions using generating functions: their critical points are in bijection with the fixed points of the diffeomorphisms they represent. From infinite dimensional Floer theory, we pass to Morse theory on a non-compact space.

A work of Patrice Le Calvez [44] shows that, for a specific choice of generating function, linking numbers of orbits still increase along the Morse differential. We define the self-linking number of a fixed point in a way that is consistent with the linking numbers of all the possible pairs of orbits, to obtain a filtration on the tensor product complex. Such filtration also exists for other generating functions, and we prove its existence using a theorem of uniqueness for generating functions due to Viterbo. After this, we go back to the Floer setting and study the behaviour of the filtration we defined under the higher homology operations between Floer complexes. The tool here is explicitly the positivity of intersections between holomorphic curves, which in the more general setting of punctured holomorphic curves in 4-dimensional symplectisations is translated into the positivity of the Siefring intersection product. The main element of novelty is given by the structural results we obtain via generating functions. We report that in the literature similar filtrations have already been observed, for instance see [35], [55] and [54], but that these works use an approach based on asymptotic analysis of pseudoholomorphic curves, which is the backbone of the definition of the Siefring product. We expect that in the future the filtration we construct here will bear dynamical applications, possibly using its Floer-theoretical definition.

The positivity of intersections between holomorphic curves appears in the paragraph "Braids and Hofer estimates" as well, hidden in the definition of the action for generators of the complex. The theory we describe is a Lagrangian intersection Floer homology in the symmetric product. It also admits a variational principle, and critical points of the action functional now are (capped) paths between a Lagrangian torus in the symmetric product to itself. The differential will be defined by counts of perturbed pseudoholomorphic discs with Lagrangian boundary conditions in the symmetric product. The positivity of intersection we have here is between the pseudoholomorphic discs and a holomorphic divisor[6] of (real) codimension 2. This phenomenon is crucial in the proof of the existence of an action filtration in our Floer theory, and in the monotonicity of the action along Floer continuation maps. The specific shape the action functional takes in this setting, together with Hofer Lipschitz properties of spectral invariants of this Floer theory, allows us to perform Hofer measurements based on braid type of Hamiltonian diffeomorphism in a specific class, which we define below. The result is an estimate of the Hofer norm of Hamiltonian diffeomorphisms based on the complexity of the braid they draw: loosely speaking, the more complex the braid is, the more energy it takes to draw it.

0.2 Main results and sketch of the proofs

Braids and Hofer estimates The content of this section will be presented in Chapter 2.

We work on an arbitrary compact surface $\Sigma_{g,p}$ with genus $g \geq 0$ and $p \geq 1$ boundary components. A collection $\underline{L}$ of $k+g$ circles is said to be pre-monotone if it satisfies certain geometric conditions (detailed in Definition 2.2.1): we are interested in the group of compactly supported Hamiltonian diffeomorphisms that fix $\underline{L}$ as a set. To every such diffeomorphism φ we may associate an element $b(\varphi)$ of the braid group of the surface. It turns out, as we shall see in Theorem 2.2.3, that the Hofer norm of φ may be estimated from below by a function that encodes the complexity of the braid $b(\varphi)$. From now on in this introduction we assume that $\Sigma_{g,p} = \mathbb{D}$, since the main result is considerably easier to state in this setting.

Theorem 0.2.1. *If $\underline{L}$ is pre monotone in the unit disc $\mathbb{D}$, and if φ fixes $\underline{L}$, then*

$$\|\varphi\| \geq c(\underline{L})|\mathrm{lk}(b(\varphi))|$$

where $c(\underline{L})$ is a constant which only depends on $\underline{L}$.

The definition of the invariant lk, the "linking number", may be found in Section 1.1, and the one of the Hofer norm in Section 1.2. An analogous result for general surfaces with boundary still holds, and its proof is contained in the above-mentioned joint work with Trifa [52]. The statement in full generality may be found in Theorem 2.2.3.

[6] The discs counted by the differential are not really pseudoholomorphic, because of a Hamiltonian perturbation term. In the construction of the homology theory however one has to change the Hamiltonian to a time-dependent constant in a neighbourhod of this divisor, thus forcing holomorphicity of the discs there.

A corollary of the Theorem is that we may estimate from below the distance between Hamiltonian diffeomorphisms preserving a common $\underline{L}$ in terms of their braid types. We use this to define, following an idea of Frédéric Le Roux, a family of norms on the braid group (or a proper subgroup, when necessary), for which we provide estimates from below.

We wish to highlight that it is the first time, at least at the best of our knowledge, that this a Hofer estimate of this kind is given without relying on non-trivial homotopy of the surface. It is indeed possible to see our results as a generalisation of those in [46, Section 1.2] and of [41, Theorem 1]. In the former, Le Roux considers the standard annulus

$$\mathbb{A} := (0, 1) \times \mathbb{S}^1$$

and a smaller one

$$A := (a, b) \times \mathbb{S}^1 \subset \mathbb{A},\ 0 < a < b < 1$$

The group of compactly supported Hamiltonian diffeomorphisms of $\mathbb{A}$ preserving A admits a morphism to $\mathbb{Z}$, the rotation number *rot*. This function measures how many times A turns around following the generator of the homotopy of $\mathbb{A}$, along any compactly supported isotopy between the identity and φ. Using the Energy-Capacity inequality in the universal covering, Le Roux then finds a lower bound for the Hofer norm of such a diffeomorphism in terms of the rotation number only:

$$\|\varphi\| \geq \mathcal{O}\left(|rot(\varphi)|\right)$$

Khanevsky then, answering to a question of Le Roux's from [46], considers a non displaceable disc D in $\mathbb{A}$ and provides a solution to an analogous problem: to each compactly supported Hamiltonian diffeomorphism φ of $\mathbb{A}$ such that $\varphi(D) = D$ we associate its rotation number $rot(\varphi)$. In this larger context, Khanevsky proves a similar inequality as above, using the Entov-Polterovich quasimorphisms defined in [22].

Let us now describe the strategy of the proof of our result on the disc. The proof for general compact surfaces with boundary relies on roughly the same idea, but with some extra technical difficulties to overcome.

The homology theory constructed by the authors in [21] is an invariant of a collection of circles $\underline{L}$ which has to satisfy certain geometric assumptions, and of a Hamiltonian diffeomorphism φ. In particular, Hofer-Lipschitz spectral invariants

$$c_{\underline{L}}(\varphi) \in \mathbb{R}$$

are defined. We embed symplectically our discs $\mathbb{D}$ into spheres $\mathbb{S}^2(1 + s)$ of different areas. Each embedding gives rise to a different homology theory, and with it different Hofer-Lipschitz spectral invariants. We cannot compute the spectral invariants directly, as often is the case outside very specific examples, but can instead compute their difference. In order to provide a more precise description of the proof, we need to give a basic description of the Floer complex.

Let $\underline{L}$ be a pre-monotone collection of k circles in $\mathbb{D}$, and denote by $\underline{L}_s$ its image in $\mathbb{S}^2(1+s)$. We consider the symmetric product $\mathrm{Sym}^k\mathbb{S}^2(1+s)$: it can be given the structure of a symplectic manifold. The collection $\underline{L}_s$ defines a Lagrangian torus

$$\mathrm{Sym}^k\underline{L}_s$$

in the quotient, and for s in a certain interval this torus is monotone. Given a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}_c(\mathbb{D})$ generated by a Hamiltonian H, denote by φ_s the extension by the identity to $\mathbb{S}^2$ and by H_s the extension of H by 0. The complex

$$CF(H_s, \underline{L}_s)$$

we are interested in is generated by capped Hamiltonian paths from $\mathrm{Sym}^k\underline{L}_s$ to itself. A capping is a homotopy between such Hamiltonian path and the constant path at a point in $\mathrm{Sym}^k\underline{L}_s$. The differential will be defined counting holomorphic discs in $\mathrm{Sym}^k\mathbb{S}^2(1+s)$ with Lagrangian boundary conditions. The homology, as per [21, Lemma 6.6], is well defined, and by [21, Lemma 6.10] it is non zero: to define spectral invariants, we now only need to have an action filtration. We now recall from [21] that there are two monotonicity constants associated to $\mathrm{Sym}^k\underline{L}_s$, a strictly positive $\lambda > 0$ and a positive or null $\eta \geq 0$. Denote now by Δ the fat diagonal in the symmetric product:

$$\Delta := \left\{ [x_1, \dots, x_k] \in \mathrm{Sym}^k\mathbb{S}^2(1+s) | \exists i \neq j, x_i = x_j \right\}$$

and let $\hat{y}$ be a generator of the Floer complex. Its action is defined to be

$$\mathcal{A}^\eta_H(\hat{y}) := \int_0^1 \mathrm{Sym}H_t(y(t))\, dt - \int_{[0,1]\times[0,1]} \hat{y}^*\omega_X - \eta[\hat{y}] \cdot \Delta$$

It is also shown in [21] that changing capping shifts the action by an integer multiple of λ, and that the action is monotone along the differential (strictly increasing or decreasing depending on the conventions).

We can now explain how to compute the difference of spectral invariants announced above. Let us consider s_1 and s_2 parameters for our embeddings. The Floer complexes

$$CF(H_{s_1}, \underline{L}_{s_2}),\ CF(H_{s_1}, \underline{L}_{s_2})$$

are in fact isomorphic as chain complexes: to prove this point, choose any biholomorphism

$$\mathbb{S}^2(1+s_1) \to \mathbb{S}^2(1+s_2)$$

and it will descend to a biholomorphism

$$\mathrm{Sym}^k\mathbb{S}^2(1+s_1) \to \mathrm{Sym}^k\mathbb{S}^2(1+s_2)$$

This biholomorphism induces a chain complex isomorphism, as it is easy to see. This chain complex isomorphism will not however respect the action filtration, but it turns out that it will simply shift it uniformly. In order to show this we use an interesting consequence of the definition of pre-monotonicity: for embeddings in $\mathbb{S}^2(1+s_i)$ with different area parameter s, the two constants λ_{s_i} associated to $\mathrm{Sym}^k(\underline{L}_{s_i})$ will coincide, while the two η_{s_i} will be different. Now, we mention that the spectral invariants $c_{\underline{L}_s}(\varphi_s)$ satisfy a (Spectrality) axiom:

$$\exists \hat{y} \in CF(H_s, \underline{L}_s) \text{ such that } \mathcal{A}^{\eta_s}_{H_s}(\hat{y}) = c_{\underline{L}_s}(\varphi_s)$$

This implies that the difference of spectral invariants will be equal to this action shift. Because this action shift is uniform, moreover, we may choose any pair of generators related by the isomorphism to compute it. Another property we obtain is that the difference we want to compute

$$c_{\underline{L}_{s_1}}(\varphi_{s_1}) - c_{\underline{L}_{s_2}}(\varphi_{s_2})$$

is additive under concatenation of Hamiltonian diffeomorphisms preserving $\underline{L}$. Because of this last property, we can focus our attention on Hamiltonian diffeomorphisms of particularly simple braid type, generators of $\mathcal{B}_k$ for instance. Consider a pair of capped Hamiltonian paths $\hat{y}_i$ from $\mathrm{Sym}^k(\underline{L}_{s_i})$ to themselves related by this isomorphism, and we assume that the two cappings, up to homotopy, are entirely contained in

$$\mathrm{Sym}^k(\mathbb{D}) \subset \mathrm{Sym}^k(\mathbb{S}^2(1+s_i))$$

Recall the definition of the action given above: in this context we easily see that

$$c_{\underline{L}_{s_1}}(\varphi_{s_1}) - c_{\underline{L}_{s_2}}(\varphi_{s_2}) = \mathcal{A}^{\eta_{s_1}}_{H_{s_1}}(\hat{y}_1) - \mathcal{A}^{\eta_{s_2}}_{H_{s_2}}(\hat{y}_2) = (\eta_{s_2} - \eta_{s_1})[\hat{y}_1] \cdot \Delta$$

Such capping are homotopies between the braid type of φ, $b(\varphi)$, and the trivial braid: if we can compute the term $[\hat{y}_1] \cdot \Delta$ from above we are done. This computation is carried out in Section 2.4.2, after describing a class of intersections of cappings with Δ which are guaranteed to be transverse. Given the Hofer Lipschitz property of the difference of spectral invariants, the intersection number we find (up to multiplication by a known constant) bounds from below the Hofer norm of φ.

A linking filtration The content of this section is the object of Chapter 3.

Let us consider a compactly supported Hamiltonian diffeomorphism φ of $\mathbb{R}^2$. Let H be any Hamiltonian generating it. Any diffeomorphism of this kind admits a generating function

$$h : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$$

which is quadratic at infinity. After perturbing the generating function, for generic g close to the standard metric on $\mathbb{R}^2 \times \mathbb{R}^k$, the Morse complex $CM(h, g; \mathbb{Z})$ is defined. The generator of the Morse complex being critical points of h, they are in bijection with the set of fixed points of φ. It is known that the homology of this complex is isomorphic to the homology of the Floer complex $CM(H, J; \mathbb{Z})$ of φ, defined for a generic choice of almost complex structure J. The Floer complex is generated by fixed points of φ, and its differential is defined counting cylinders

$$u : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$$

satisfying the Floer equation

$$\partial_s u + J(\partial_t u - X_H(u)) = 0$$

It is a classical fact that the graph of u,

$$\overline{u} : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^2$$

is pseudoholomorphic, and pseudoholomorphic curves in 4-manifolds intersect positively. Consider now four distinct fixed points $x_\pm, y_\pm$ of φ, such that there are two Floer cylinders, u and v, asymptotic to these generators:

$$\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm, \ \lim_{s \to \pm\infty} v(s, \cdot) = y_\pm$$

Seeing the pair of cylinders (u, v) as a homotopy from the braid (x_-, y_-) to the braid (x_+, y_+), and using the positivity of intersections of holomorphic curves we conclude that

$$\mathrm{lk}(x_-, y_-) \leq \mathrm{lk}(x_+, y_+)$$

It is possible to obtain a similar result using generators functions: Patrice Le Calvez in [44] proves it decomposing φ as a product of twist maps, and constructing a generating function for which the linking number between orbits has a Lyapunov property. Using this and a dominated splitting of $\mathbb{R}^2 \times \mathbb{R}^k$ (still defined in [44]), we prove the following:

Theorem 0.2.2. *Let φ a compactly supported Hamiltonian diffeomorphism of the plane with its standard symplectic form $dx \wedge dy$. Then for any generating function quadratic at infinity $S : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ for φ, there exists a non-degenerate quadratic form Q on $\mathbb{R}^l$, a Riemannian metric g on $\mathbb{R}^{2+k+l}$ such that the pair $(S \oplus Q, g)$ is both Palais-Smale and Morse-Smale, and making (an extension of) the function*

$$I : CM(S \oplus Q, g; \mathbb{Z}) \otimes CM(S \oplus Q, g; \mathbb{Z}) \to \mathbb{Z},$$

$$I(p \otimes q) := \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & p \neq q \\ -\left\lceil \frac{CZ(\gamma_p)}{2} \right\rceil & p = q \end{cases}$$

into an increasing filtration of the tensor complex.

The number $CZ(\gamma_p)$ in the Theorem is the Conley-Zehnder index of γ_p, which coincides, up to translation, with the Morse index of p. The multiplicative factor of $\frac{1}{2}$ appears in light of our normalisation of the linking number.

The proof of the Theorem is divided into two parts: we show the existence of this kind of filtration for generating functions of Le Calvez type, and then push it forward to all other generating functions.

The proof on the generating functions defined by Calvez relies on the announced Lyapunov property for the linking number, and on the study of the asymptotics of different gradient lines γ_1, γ_2 approaching the same critical point x. There is in fact an interplay between the dominated splitting at x and the asymptotic linking of curves represented by points $\gamma_1(t)$ and $\gamma_2(t)$ for $t \gg 0$ (it is not clear for general generating functions how points in the vector bundle $\mathbb{R}^2 \times \mathbb{R}^k$ are associated to loops in $\mathbb{R}^2$, but the construction that Le Calvez uses makes this relation apparent).

Once we obtain the existence of the filtration for a Le Calvez generating function, we extend it to arbitrary generating functions using Viterbo's Uniqueness Theorem [73], [69]. In order to apply it, we have to show that the ones defined by Le Calvez are generating functions quadratic at infinity in the classical sense, up to gauge equivalence: this is one of the three elementary operations on generating functions as defined by Viterbo, and it induces an isomorphism of Morse complexes. Filtrations may also be pushed forward along these elementary operations, so that an application of the Uniqueness Theorem then proves the main result.

We prove a folklore theorem about the relation between the Hofer metric on $\mathrm{Ham}_c(\mathbb{R}^2)$ and the supremum norm of generating functions. If two diffeomorphisms φ, ψ are Hofer-close, then there are two generating functions, one for φ and one for ψ, which are $\mathscr{C}^0$-close: this is the content of Lemma 3.4.5. We also show that the filtration I defined above increases along continuation maps between generating functions. With these elements we are able to prove a simple case of [3, Theorem 2]: for small Hofer perturbations, linking numbers of orbits persists (see Proposition 3.4.6).

We now go back to the Floer perspective: let φ be a Hamiltonian diffeomorphism of a closed oriented surface Σ_g of genus $g \geq 1$ or of $\mathbb{R}^2$ (in which case we also assume φ is compactly supported). We denote by Σ the surface on which φ is defined. Some classical results of Hofer, Wysocki and Zehnder show that, near critical points of the action functional of φ, there is an infinite dimensional analogue of the dominated splitting defined by Le Calvez. Choose a fixed point x of φ. They show, as a start, that each eigenfunction of the linearised Floer operator at x has a well-defined linking number with respect to the x itself. If two different eigenfunctions share the same eigenvalue moreover, then they have the same linking number with x. For each integer n, the space of eigenfunctions having linking number n with x has dimension 2, and the function associating to an eigenvalue the corresponding linking number is monotone increasing.

Much like in the Morse case, a cylinder

$$u : \mathbb{R} \times \mathbb{S}^1 \to \Sigma$$

negatively (resp. positively) asymptotic to a fixed point x will locally be represented by a sum of eigenfunctions with positive (resp. negative) eigenvalues, and by the above facts about eigenfunctions and linking numbers there are a priori bounds on the linking number between the curves

$$u_s : \mathbb{S}^1 \to \Sigma,\ t \mapsto u(s,t) \text{ and } x : \mathbb{S}^1 \to \Sigma, t \mapsto x(t)$$

for $s \to -\infty$ (resp. $s \to +\infty$). These estimates are used to define the Siefring product between Floer cylinders, and its positivity gives us the Floer analogue of the main result we have in the Morse setting:

Theorem 0.2.3. *Let $\varphi \in Ham(\Sigma)$ (compactly supported if $\Sigma = \mathbb{R}^2$) be non degenerate (if φ is compactly supported, we only require non degeneracy on the interior of the support) and generated by a Hamiltonian H. Then for generic almost complex structure J, we have an increasing filtration I on*

$$CM(H, J; \mathbb{Z}) \otimes CM(H, J; \mathbb{Z})$$

defined on generators by

$$I(x \otimes y) := \begin{cases} \frac{1}{2}\mathrm{lk}(x,y) & x \neq y \\ -\left\lceil \frac{CZ(x)}{2} \right\rceil & x = y \end{cases}$$

We immediately see that the filtrations obtained via Floer and Morse theoretical methods, perhaps unsurprisingly, coincide.

We finish Chapter 3 studying how the filtration I behaves with respect to the pair of pants product in Floer homology. We define the pair of pants with $p+1$ legs

$$S_p := \mathbb{S}^2 \setminus \{a_1, \ldots, a_{p+1}\} \text{ with } i \neq j \Rightarrow a_i \neq a_j$$

as the two dimensional sphere with $p+1$ punctures, of which one positive and the remaining p ones negative. The pair of pants product is a map

$$\mathcal{P}_p : CF(H, J; \mathbb{Z})^{\otimes p} \to CF(H^{\#p}, J; \mathbb{Z})$$

that to p fixed points of φ associates a fixed point of φ^p. Remark that $CF(H^{\#p}, J; \mathbb{Z})^{\otimes 2}$ may be endowed with the filtration I as above, while we may define I_p on the source space of $\mathcal{P}_p$:

$$I_p((x_1^1 \otimes x_p^1) \otimes (x_1^2 \otimes x_p^2)) := \sum_{j=1}^{p} I(x_j^1 \otimes x_j^2)$$

The operation $\mathcal{P}_p$ is defined counting curves

$$u : S_p \to \Sigma$$

which are pseudoholomorphic far from the punctures, and which satisfy Floer equation near them. Their graphs

$$\bar{u} : S_p \to S_p \times \Sigma$$

in this case as well turns out to be pseudoholomorphic. We aim to show the following

Theorem 0.2.4. *For $x_1^1, \dots, x_p^1, x_1^2, \dots, x_p^2$ fixed points of φ, we have*

$$I_p((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) \leq I(\mathcal{P}_p(x_1^1 \otimes \cdots \otimes x_p^1) \otimes \mathcal{P}_p(x_1^2 \otimes \cdots \otimes x_p^2))$$

To prove this Theorem we are lead to compute Siefring intersection products $\bar{u} * \bar{v}$ between graphs of pairs of pants u, v involved in the definition of $\mathcal{P}_p$. We need to distinguish two possibilities: either $u = v$ or $u \neq v$. The former case is very simple to deal with: a simple degree count is enough, because both I and I_p are functions of Conley-Zehnder indices of the fixed points $x_1^1, \dots, x_p^1, x_1^2, \dots, x_p^2$. In the other case, it turns out that $\bar{u} * \bar{v}$ bounds from below the difference

$$I_p((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) - I(\mathcal{P}_p(x_1^1 \otimes \cdots \otimes x_p^1) \otimes \mathcal{P}_p(x_1^2 \otimes \cdots \otimes x_p^2))$$

so that it is enough to prove $\bar{u} * \bar{v} \geq 0$. Pseudoholomorphicity is enough to conclude this, and proves the Theorem.

Introduction française

Dans cette thèse, nous étudions comment nous pouvons extraire des informations sur les difféomorphismes hamiltoniens de surfaces en utilisant des tresses obtenues à partir de trajectoires hamiltoniennes. Nous allons aborder ce problème depuis deux points de vue très différents : un point de vue global, dans lequel nous utilisons les tresses pour trouver des estimations de la distance d'Hofer entre les difféomorphismes d'une classe donnée, et un autre, local, dans lequel nous nous concentrons sur un difféomorphisme hamiltonien spécifique. Les outils que nous utilisons sont un variant de l'homologie de Floer lagrangianne, récemment défini par Cristofaro-Gardiner, Humilière, Mak, Seyfaddini et Smith dans [21] pour le premier point de vue, et pour le deuxième les fonctions génératrices pour les difféomorphismes hamiltoniens (avec ses relations avec l'homologie de Floer hamiltonienne). Les résultats contenus ici proviendront de [51], d'un travail avec Ibrahim Trifa [52], et d'un travail pas encore publié de l'auteur.

0.3 Un aperçu: contexte et méthodes

Le groupe de tresses d'une surface est le groupe fondamental de son espace de configuration. Soit $\Sigma_{g,p}$ différente de la sphère, avec genre $g \geq 0$ et $p \geq 0$ composantes de bord. Considérons l'action naturelle du groupe des permutations sur n lettres $\mathfrak{S}_n$ sur l'n-ième puissance cartésienne de $\Sigma_{g,p}$ privée de la grande diagonale

$$\tilde{\Delta} = \{ (x_1, \ldots, x_n) | \exists i \neq j, x_i = x_j \}$$

Le quotient est appelé n-ième espace de configuration, et nous pouvons définir le groupe de tresses $\mathcal{B}_{n,g,p}$ par

$$\mathcal{B}_{n,g,p} := \pi_1 \left((\Sigma_{g,p}^n \setminus \tilde{\Delta}) / \mathfrak{S}_n \right)$$

Lorsque $g = 0, p = 1$, c'est-à-dire dans le cas du disque, on retrouve le groupe de tresses habituel, décrit par la présentation classique

$$\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i : |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} : 1 \leqslant i \leqslant n-2 \end{array} \right\rangle$$

Le nombre d'enlacement d'une tresse, également appelé "exposant" dans la littérature de la topologie de la basse dimension, est l'homomorphisme de groupes

$$\mathcal{B}_{n,0,1} \to \mathbb{Z}$$

qui envoie les générateurs sur 1.

Dans ce travail, nous étudions les difféomorphismes hamiltoniens en utilisant les tresses formées par leurs orbites. Un difféomorphisme hamiltonien est le flot au temps 1 d'un champ de vecteurs hamiltonien, qui à son tour est un champ de vecteurs de gradient d'une fonction par rapport à une forme symplectique. En définitive, les nombres d'enlacement contiennent des informations sur le difféomorphisme hamiltonien en raison d'un principe variationnel décrivant ses points fixes. Ce principe stipule que les points critiques de la fonctionnelle d'action, définie sur l'espace des lacets d'une variété symplectique, sont précisément des orbites hamiltoniennes fermés. Une théorie de type Morse peut être construite pour la fonctionnelle d'action : c'est ainsi, philosophiquement, que la théorie de Floer est définie. Les générateurs des groupes d'homologie de Floer sont alors des lacets hamiltoniennes fermés, et la différentielle est calculée en comptant des cylindres pseudoholomorphes perturbés avec des asymptotiques aux lacets hamiltoniens. Il s'avère que les courbes pseudo-holomorphes (dans notre cas, les graphes des courbes de Floer) dans les variétés de dimension 4 s'intersectent positivement, et ce fait admet une interprétation du point de vue de la théorie des tresses : les nombres d'enlacement des tresses formées par les orbites hamiltoniennes augmentent le long de la différentielle de Floer.

Dans la direction de recherche décrite dans le paragraphe "Une filtration en enlacements" ci-dessous, nous discrétisons le problème variationnel en dimension finie en utilisant les fonctions génératrices : leurs points critiques sont en bijection avec les points fixes des difféomorphismes qu'elles représentent. De la théorie de Floer, définie en dimension infinie, nous passons à la théorie de Morse sur un espace non compact mais fini-dimensionnel.

Un travail de Patrice Le Calvez [44] montre que, pour un choix spécifique de fonction génératrice, les nombres d'enlacement des orbites augmentent toujours le long de la différentielle de Morse. Nous définissons le nombre d'auto-enlacement d'un point fixe d'une manière qui est cohérente avec les nombres d'enlacement de toutes les paires d'orbites possibles, afin d'obtenir une filtration sur le complexe produit tensoriel. Une telle filtration existe aussi pour d'autres fonctions génératrices, et nous prouvons son existence en utilisant un théorème d'unicité des fonctions génératrices dû à Viterbo. Après cela, nous revenons au cadre de la théorie de Floer et étudions le comportement de la filtration que nous avons définie sous les opérations de produit homologique entre les complexes de Floer. L'outil utilisé ici est explicitement la positivité des intersections entre courbes holomorphes, qui, dans le cadre plus général des courbes holomorphes piqûrées dans les symplectisations de dimension 4, se traduit dans la positivité du produit d'intersection de Siefring.

Le principal élément de nouveauté réside dans les résultats structurels que nous obtenons à travers des fonctions génératrices. Nous signalons que des filtrations similaires ont déjà été observées dans la littérature, voir par exemple [35], [55] et [54], mais que ces travaux utilisent plutôt une approche basée sur l'analyse asymptotique des courbes pseudo-holomorphes, qui est l'épine dorsale de la définition du produit de Siefring. Nous nous attendons qu'à l'avenir la filtration que nous construisons ici aura des applications dynamiques, peut-être en utilisant sa définition en homologie de Floer.

La positivité des intersections entre courbes holomorphes apparaît également dans le paragraphe "Tresses et estimations de norme d'Hofer", cachée dans la définition de l'action des générateurs du complexe. La théorie que nous décrivons est une homologie de Floer lagrangienne dans le produit symétrique. Elle admet également un principe variationnel, et les points critiques de la fonctionnelle d'action sont maintenant des chemins (avec capping) dans le produit symétrique entre un tore lagrangien et lui-même. La différentielle sera définie par un comptage de disques pseudo-holomorphes perturbés, avec conditions lagrangiennes au bord, dans le produit symétrique. La positivité des intersections que nous avons ici est entre les disques pseudoholomorphes et un diviseur holomorphe [7] de codimension (réelle) 2. Ce phénomène est crucial dans la preuve de l'existence de la filtration d'action dans notre théorie de Floer, et dans la monotonicité de l'action le long des applications de continuation de Floer. La forme spécifique que prend la fonctionnelle d'action dans ce cadre, ainsi que les propriétés de Hofer Lipschitz des invariants spectraux de cette théorie de Floer, nous permettent d'effectuer des mesures d'énergie d'Hofer basées sur le type de tresse du difféomorphisme hamiltonien dans une classe spécifique, que nous définissons ci-dessous. Le résultat est une estimation de la norme d'Hofer des difféomorphismes hamiltoniens basée sur la complexité de la tresse qu'ils dessinent : en gros, plus la tresse est complexe, plus il faut d'énergie pour la dessiner.

0.4 Les résultats principaux et leurs preuves

Tresses et estimations de norme d'Hofer Le contenu de cette section sera présenté au chapitre 2.

Nous travaillons sur une surface compacte arbitraire $\Sigma_{g,p}$ de genre $g \geq 0$ et $p \geq 1$ composantes de bord. Une collection $\underline{L}$ de $k+g$ cercles est dite prémonotone si elle satisfait certaines conditions géométriques (détaillées dans la Définition 2.2.1) : nous nous intéressons au groupe de difféomorphismes hamiltoniens à support compact qui fixent $\underline{L}$ en tant qu'ensemble. A chaque difféomorphisme φ de ce type nous pouvons associer un élément $b(\varphi)$ du groupe de

[7] Les disques comptés par la différentielle ne sont pas vraiment pseudoholomorphes, à cause d'un terme de perturbation hamiltonienne. Dans la construction de la théorie de l'homologie, il faut cependant changer l'hamiltonien en une constante dépendante du temps dans un voisinage de ce diviseur, forçant ainsi l'holomorphie des disques à cet endroit.

tresses de la surface $\mathcal{B}_{k,g,p}$. Il s'avère, comme nous le verrons dans le théorème 2.2.3, que la norme de Hofer de φ peut être bornée du bas par une fonction qui encode la complexité de la tresse $b(\varphi)$. L'énoncé pour le disque est le plus simple, et nous le rapportons ci-dessous :

Theorem 0.4.1. *Si $\underline{L}$ est pré-monotone dans le disque standard $\mathbb{D}$, et si φ fixe $\underline{L}$, alors*

$$\|\varphi\| \geq c(\underline{L})|\mathrm{lk}(b(\varphi))|$$

où $c(\underline{L})$ est une constante qui ne dépend que de $\underline{L}$.

La définition de l'invariant lk, le "nombre d'enlacement" se trouve en Section 1.1, et celle de la norme d'Hofer en Section 1.2. Le résultat pour des surfaces à bord plus générales est bien plus compliqué à énoncer, essentiellement parce que nous autorisons un nombre quelconque de composantes de bord pour la surface. La preuve est contenue dans le travail avec Trifa [52], et l'énoncé en toute généralité se trouve dans le théorème 2.2.3.

Un corollaire du théorème est que l'on peut estimer par le bas la distance entre les difféomorphismes hamiltoniens préservant une configuration pré-monotone commune $\underline{L}$ en termes de leurs types de tresses. Nous utilisons ceci pour définir, en suivant une idée de Frédéric Le Roux, une famille de normes sur le groupe de tresses (ou un sous-groupe propre, si nécessaire), pour lesquelles nous fournissons des estimations par le bas.

Nous souhaitons souligner que c'est la première fois, du moins à notre connaissance, qu'une telle estimation d'énergie d'Hofer est donnée sans s'appuyer sur une homotopie non triviale de la surface. Il est en effet possible de considérer nos résultats comme une généralisation de ceux de [46, Section 1.2] et de [41, Théorème 1]. Dans le premier, Le Roux considère l'anneau standard

$$\mathbb{A} := (0,1) \times \mathbb{S}^1$$

et un anneau plus petit

$$A := (a,b) \times \mathbb{S}^1 \subset \mathbb{A},\ 0 < a < b < 1$$

Le groupe des difféomorphismes hamiltoniens à support compact de $\mathbb{A}$ préservant A admet un morphisme vers $\mathbb{Z}$, le nombre de rotation rot. Cette fonction mesure le nombre de fois que A tourne autour du générateur de l'homotopie de $\mathbb{A}$, le long de n'importe quelle isotopie à support compact entre l'identité et φ. En utilisant l'inégalité capacité-énergie dans le revêtement universel, Le Roux trouve ensuite une borne inférieure pour la norme de Hofer d'un tel difféomorphisme en termes de nombre de rotation seulement :

$$\|\varphi\| \geq \mathcal{O}\left(|rot(\varphi)|\right)$$

Khanevsky, répondant à une question de Le Roux dans [46], considère un disque non déplaçable D dans $\mathbb{A}$ et fournit une solution à un problème analogue : à chaque difféomorphisme hamiltonien compactement supporté φ de $\mathbb{A}$ tel que

$\varphi(D) = D$, nous associons son nombre de rotation $rot(\varphi)$. Dans ce contexte plus large, Khanevsky prouve une inégalité similaire à la précédente, en utilisant les quasimorphismes d'Entov-Polterovich définis dans [22].

Décrivons maintenant la stratégie de la preuve de notre résultat sur le disque. La démonstration pour les surfaces compactes à bord générales repose grosso modo sur la même idée, mais il y a quelques difficultés techniques supplémentaires à surmonter.

La théorie homologique construite par les auteurs dans [21] est un invariant d'une collection de cercles $\underline{L}$ qui doit satisfaire certaines hypothèses géométriques, et d'un difféomorphisme hamiltonien φ. En particulier, les invariants spectraux

$$c_{\underline{L}}(\varphi) \in \mathbb{R}$$

sont définis, et ils ont une propriété de lipschitzianité par rapport à la norme de Hofer. Nous plongeons symplectiquement nos disques $\mathbb{D}$ dans des sphères $\mathbb{S}^2(1+s)$ de différentes aires. Chaque plongement donne lieu à une théorie homologique différente, et avec elle des invariants spectraux Hofer-Lipschitz différents. Nous ne pouvons pas calculer directement les invariants spectraux, comme c'est souvent le cas en dehors d'exemples très spécifiques, mais nous pouvons calculer leur différence. Afin de fournir une description plus précise de la preuve, nous devons donner une description de base du complexe de Floer.

Soit $\underline{L}$ une collection pré-monotone de k cercles dans $\mathbb{D}$, et désignons par $\underline{L}_s$ son image dans $\mathbb{S}^2(1+s)$. Nous considérons le produit symétrique $\mathrm{Sym}^k\mathbb{S}^2(1+s)$: on peut lui donner une structure de variété symplectique. La collection $\underline{L}_s$ définit un tore lagrangien

$$\mathrm{Sym}^k\underline{L}_s$$

dans le quotient, et pour s dans un certain intervalle ce tore est monotone. Etant donné un difféomorphisme hamiltonien $\varphi \in \mathrm{Ham}_c(\mathbb{D})$ généré par un hamiltonien H, notons par φ_s l'extension par l'identité à $\mathbb{S}^2$ et par H_s l'extension de H par 0 à la sphère. Le complexe

$$CF(H_s, \underline{L}_s)$$

qui nous intéresse est engendré par des chemins hamiltoniens avec capping de $\mathrm{Sym}^k\underline{L}_s$ à lui-même. Un capping est une homotopie entre un tel chemin hamiltonien et le chemin constant en un point de $\mathrm{Sym}^k\underline{L}_s$. La différentielle sera définie en comptant les disques holomorphes dans $\mathrm{Sym}^k\mathbb{S}^2(1+s)$ avec des conditions lagrangiennes aux bords. L'homologie, selon [21, Lemma 6.6], est bien définie, et par [21, Lemma 6.10] elle est non nulle : pour définir des invariants spectraux, nous avons maintenant seulement besoin d'avoir une filtration d'action. Nous rappelons maintenant, suivant [21], qu'il existe deux constantes de monotonie associées à $\mathrm{Sym}^k\underline{L}_s$, un $\lambda > 0$ strictement positif et un $\eta \geq 0$ positif ou nul. Soit maintenant Δ la grosse diagonale du produit symétrique :

$$\Delta := \left\{ [x_1, \ldots, x_k] \in \mathrm{Sym}^k\mathbb{S}^2(1+s) | \exists i \neq j, x_i = x_j \right\}$$

Soit $\hat{y}$ un générateur du complexe de Floer : son action est définie comme étant

$$\mathcal{A}^{\eta}_{H}(\hat{y}) := \int_0^1 \mathrm{Sym} H_t(y(t))\, dt - \int_{[0,1]\times[0,1]} \hat{y}^*\omega_X - \eta[\hat{y}] \cdot \Delta$$

On montre également dans [21] que le changement de capping change l'action d'un multiple entier de λ, et que l'action est monotone le long de la différentielle (strictement croissante ou décroissante selon les conventions).

Nous pouvons maintenant expliquer comment calculer la différence des invariants spectraux annoncée ci-dessus. Considérons les paramètres s_1 et s_2 pour nos plongements. Les complexes de Floer

$$CF(H_{s_1}, \underline{L}_{s_2}),\ CF(H_{s_1}, \underline{L}_{s_2})$$

sont en fait isomorphes en tant que complexes de chaînes. Pour prouver ce point, choisissons n'importe quel biholomorphisme

$$\mathbb{S}^2(1+s_1) \to \mathbb{S}^2(1+s_2)$$

et il descendra à un biholomorphisme

$$\mathrm{Sym}^k \mathbb{S}^2(1+s_1) \to \mathrm{Sym}^k \mathbb{S}^2(1+s_2)$$

Ce biholomorphisme induit un isomorphisme de complexes de chaînes, comme l'on peut aisément montrer. Cet isomorphisme de complexes de chaînes ne respectera cependant pas la filtration d'action, mais il la changera de façon uniforme. Pour montrer cela, nous utilisons une conséquence intéressante de la définition de la pré-monotonicité : pour des plongements dans $\mathbb{S}^2(1+s_i)$ avec paramètres d'aire s différents, les deux constantes λ_{s_i} associées à $\mathrm{Sym}^k(\underline{L}_{s_i})$ coïncideront, tandis que les deux η_{s_i} seront différents. Remarquons maintenant que les invariants spectraux $c_{\underline{L}_s}(\varphi_s)$ satisfont un axiome de spectralité :

$$\exists \hat{y} \in CF(H_s, \underline{L}_s) \text{ tel que } \mathcal{A}^{\eta_s}_{H_s}(\hat{y}) = c_{\underline{L}_s}(\varphi_s)$$

Cela implique que la différence des invariants spectraux sera égale à ce décalage d'action. Comme ce décalage d'action est uniforme, nous pouvons choisir n'importe quelle paire de générateurs liés par l'isomorphisme pour le calculer. Une autre propriété que nous obtenons est que la différence que nous souhaitons calculer

$$c_{\underline{L}_{s_1}}(\varphi_{s_1}) - c_{\underline{L}_{s_2}}(\varphi_{s_2})$$

est additive sous concaténation de difféomorphismes hamiltoniens préservant $\underline{L}$. Grâce à cette dernière propriété, nous pouvons concentrer notre attention sur les difféomorphismes hamiltoniens de type de tresse particulièrement simple, les générateurs de $\mathcal{B}_k$ par exemple. Considérons une paire de chemins hamiltoniens avec cappings $\hat{y}_i$ de $\mathrm{Sym}^k(\underline{L}_{s_i})$ à lui-même reliés par cet isomorphisme, et nous supposons que les deux cappings, à homotopie près, soient entièrement contenus dans

$$\mathrm{Sym}^k(\mathbb{D}) \subset \mathrm{Sym}^k(\mathbb{S}^2(1+s_i))$$

Rappelons la définition de l'action donnée ci-dessus : dans ce contexte, nous voyons facilement que

$$c_{\underline{L}_{s_1}}(\varphi_{s_1}) - c_{\underline{L}_{s_2}}(\varphi_{s_2}) = \mathcal{A}^{\eta_{s_1}}_{H_{s_1}}(\hat{y}_1) - \mathcal{A}^{\eta_{s_2}}_{H_{s_2}}(\hat{y}_2) = (\eta_{s_2} - \eta_{s_1})[\hat{y}_1] \cdot \Delta$$

De tels cappings sont des homotopies entre le type de tresse de φ, $b(\varphi)$, et la tresse triviale : si nous réussissons à calculer le terme $[\hat{y}_1] \cdot \Delta$ à partir de ce qui précède, nous avons terminé. Ce calcul est effectué dans la section 2.4.2, après avoir décrit un type d'intersections de cappings avec Δ dont la transversalité est garantie. Donnée la propriété de Hofer lipschitzianité des invariants spectraux, on arrive à montrer que le nombre d'intersection calculé (à une constante multipicative près) borne du bas la norme d'Hofer de φ.

Une filtration en enlacements Le contenu de cette section fait l'objet du chapitre 3.

Considérons un difféomorphisme hamiltonien à support compact φ de $\mathbb{R}^2$. Soit H un hamiltonien quelconque qui le génère. Tout difféomorphisme de ce type admet une fonction génératrice

$$h : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$$

qui est quadratique à l'infini. Après perturbation de la fonction génératrice, pour un g générique proche de la métrique standard sur $\mathbb{R}^2 \times \mathbb{R}^k$, le complexe de Morse $CM(h, g; \mathbb{Z})$ est défini. Les générateurs du complexe de Morse étant des points critiques de h, ils sont en bijection avec l'ensemble des points fixes de φ. On sait que l'homologie de ce complexe est isomorphe à l'homologie du complexe de Floer $CM(H, J; \mathbb{Z})$ de φ, défini pour un choix générique de structure presque complexe J. Le complexe de Floer est engendré par les points fixes de φ, et sa différentielle est définie en comptant les cylindres

$$u : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$$

qui satisfont l'équation de Floer

$$\partial_s u + J(\partial_t u - X_H(u)) = 0$$

C'est un fait classique que le graphe de u,

$$\overline{u} : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^2$$

est pseudo-holomorphe, et il est aussi connu que les courbes pseudo-holomorphes dans les 4-variétés s'intersectent positivement. Considérons maintenant quatre points fixes distincts $x_\pm, y_\pm$ de φ tels qu'il existe deux cylindres de Floer, u et v, asymptotiques à ces générateurs du complexe :

$$\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm, \ \lim_{s \to \pm\infty} v(s, \cdot) = y_\pm$$

En considérant la paire de cylindres (u, v) comme une homotopie de la tresse (x_-, y_-) vers la tresse (x_+, y_+), et en utilisant la positivité des intersections de courbes holomorphes, nous en concluons que

$$\mathrm{lk}(x_-, y_-) \leq \mathrm{lk}(x_+, y_+)$$

Il est possible d'obtenir un résultat similaire en utilisant des fonctions génératrices : Patrice Le Calvez dans [44] le prouve en décomposant φ comme un produit d'applications déviants la verticale, et en construisant une fonction génératrice pour laquelle le nombre d'enlacement entre les orbites a une propriété de Lyapunov. En utilisant ceci et une décomposition subordonnée de $\mathbb{R}^2 \times \mathbb{R}^k$ (toujours définie dans [44]), nous pouvons prouver ce qui suit :

Theorem 0.4.2. *Soit φ un difféomorphisme hamiltonien du plan à support compact avec sa forme symplectique standard $dx \wedge dy$. Alors, pour toute fonction génératrice quadratique à l'infini $S : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ pour φ, il existe une forme quadratique non dégénérée Q sur $\mathbb{R}^l$, une métrique riemannienne g sur $\mathbb{R}^{2+k+l}$ telle que la paire $(S \oplus Q, g)$ est à la fois de Palais-Smale et de Morse-Smale, et rendant (une extension de) la fonction*

$$I : CM(S \oplus Q, g; \mathbb{Z}) \otimes CM(S \oplus Q, g; \mathbb{Z}) \to \mathbb{Z},$$
$$I(p \otimes q) := \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & p \neq q \\ -\left\lceil \frac{CZ(\gamma_p)}{2} \right\rceil & p = q \end{cases}$$

une filtration croissante du complexe produit tensoriel.

Le nombre $CZ(\gamma_p)$ du théorème est l'indice de Conley-Zehnder de γ_p, qui coïncide, à translation près, avec l'indice de Morse de p. Le facteur multiplicatif de $\frac{1}{2}$ apparaît à cause de notre normalisation du nombre d'enlacement.

La preuve du théorème est divisée en deux parties : nous montrons l'existence de cette filtration pour les fonctions génératrices de type Le Calvez, puis nous la poussons en avant à toute autre fonction génératrice.

La preuve sur les fonctions génératrices définies par Calvez repose sur la propriété de Lyapunov déjà annoncée pour le nombre d'enlacement, et sur l'étude de l'asymptotique des différentes lignes de gradient γ_1, γ_2 approchant le même point critique x. Il existe en fait une interaction entre la décomposition subordonnée en x et les enlacements asymptotiques des courbes représentées par les points $\gamma_1(t)$ et $\gamma_2(t)$ pour $t \gg 0$ (il n'est pas clair pour les fonctions génératrices générales comment les points du fibré vectoriel $\mathbb{R}^2 \times \mathbb{R}^k$ sont associés à des lacets dans $\mathbb{R}^2$, mais la construction utilisée par Le Calvez rend cette relation évidente).

Une fois que nous avons obtenu l'existence de la filtration pour une fonction génératrice de Le Calvez, nous l'étendons à des fonctions génératrices arbitraires en utilisant le théorème d'unicité de Viterbo [73], [69]. Pour l'appliquer, nous devons montrer que celles définies par Le Calvez sont des fonctions génératrices quadratiques à l'infini au sens classique, à une équivalence de jauge près : c'est l'une des trois opérations élémentaires sur les fonctions génératrices définies par Viterbo, et elle induit un isomorphisme des complexes de Morse. Les filtrations peuvent aussi être poussées en avant le long de ces opérations élémentaires, de sorte qu'une application du théorème d'unicité prouve le résultat principal.

Nous prouvons un théorème folklorique sur la relation entre la métrique de Hofer sur $\mathrm{Ham}_c(\mathbb{R}^2)$ et la norme $\mathscr{C}^0$ des fonctions génératrices. Si deux difféomorphismes φ, ψ sont proches au sens d'Hofer, alors il existe deux fonctions génératrices, une pour φ et une pour ψ, qui sont proches pour la norme $\mathscr{C}^0$: c'est le contenu du lemme 3.4.5. Nous montrons aussi que la filtration I définie ci-dessus augmente le long des applications de continuation entre fonctions génératrices. Avec ces éléments, nous pouvons prouver un cas simple de [3, Théorème 2] : pour des perturbations Hofer-petites, les enlacements desorbites persistent (voir Proposition 3.4.6).

Revenons maintenant au point de vue de la théorie de Floer : soit φ un difféomorphisme hamiltonien d'une surface fermée orientée Σ_g de genre ≥ 1 ou de $\mathbb{R}^2$ (dans ce cas, nous supposons également que φ est compactement supporté). Nous désignons par Σ la surface sur laquelle φ est défini. Certains résultats classiques de Hofer, Wysocki et Zehnder [31] montrent que, près des points critiques de la fonctionnelle d'action de φ, il existe un analogue en dimension infinie de la décomposition subordonnée définie par Le Calvez. Choisissons un point fixe x de φ. Ils montrent, pour commencer, que chaque fonction propre de l'opérateur de Floer linéarisé en x possède un nombre d'enlacement bien défini par rapport à x lui-même. Si deux fonctions propres différentes partagent la même valeur propre, alors elles ont le même nombre d'enlacement avec x. Pour chaque entier n, l'espace des fonctions propres ayant un nombre d'enlacement n avec x est de dimension 2, et la fonction qui associe à une valeur propre le nombre d'enlacement correspondant est monotone croissante.

Comme dans le cas Morse, un cylindre

$$u : \mathbb{R} \times \mathbb{S}^1 \to \Sigma$$

négativement (resp. positivement) asymptotique à un point fixe x peut être localement représenté par une somme de fonctions propres aux valeurs propres positives (resp. négatives), et par les faits ci-dessus sur les fonctions propres et leur nombres d'enlacement, il y a des bornes a priori sur le nombre d'enlacement entre les courbes

$$u_s : \mathbb{S}^1 \to \Sigma,\ t \mapsto u(s,t) \text{ et } x : \mathbb{S}^1 \to \Sigma, t \mapsto x(t)$$

pour $s \to -\infty$ (resp. $s \to +\infty$). Ces bornes sont utilisées pour définir le produit de Siefring entre cylindres de Floer, et sa positivité nous donne l'analogue de Floer du résultat principal que nous avons dans le cadre de Morse :

Theorem 0.4.3. *Soit $\varphi \in Ham(\Sigma)$ (à support compact si $\Sigma = \mathbb{R}^2$) non dégénéré (si φ est à support compact, nous exigeons seulement la non dégénérescence à l'intérieur du support) et généré par un hamiltonien H. Alors, pour une* *structure presque complexe générique J, nous avons une filtration croissante I sur*

$$CM(H,J;\mathbb{Z})\otimes CM(H,J;\mathbb{Z})$$

définie sur les générateurs par

$$I(x\otimes y):=\begin{cases}\frac{1}{2}\mathrm{lk}(x,y) & x\neq y\\ -\left\lceil\frac{CZ(x)}{2}\right\rceil & x=y\end{cases}$$

Nous voyons immédiatement que les filtrations obtenues par les méthodes théoriques de Floer et de Morse coïncident, ce qui n'est peut-être pas surprenant.

Nous terminons le chapitre 3 en étudiant le comportement de la filtration I par rapport au produit paire de pantalons sur l'homologie de Floer. Nous définissons la paire de pantalons avec $p+1$ jambes

$$S_p:=\mathbb{S}^2\setminus\{a_1,\dots,a_{p+1}\}\text{ avec }i\neq j\Rightarrow a_i\neq a_j$$

comme la sphère bidimensionnelle avec $p+1$ piqûres, dont une positive et les restantes p négatives. Le produit paire de pantalons est une fonction

$$\mathcal{P}_p:CF(H,J;\mathbb{Z})^{\otimes p}\to CF(H^{\#p},J;\mathbb{Z})$$

qui à p points fixes de φ associe un point fixe de φ^p. Remarquons que $CF(H^{\#p},J;\mathbb{Z})^{\otimes 2}$ peut être equippé de la filtration I comme ci-dessus, tandis que nous pouvons définir une filtration I_p sur l'espace source de $\mathcal{P}_p$:

$$I_p((x_1^1\otimes x_p^1)\otimes(x_1^2\otimes x_p^2)):=\sum_{j=1}^{p}I(x_j^1\otimes x_j^2)$$

L'opération $\mathcal{P}_p$ est définie en comptant les courbes

$$u:S_p\to\Sigma$$

qui sont pseudo-holomorphes loin des piqûres, et qui vont satisfaire l'équation de Floer près d'elles. Leurs graphes

$$\bar{u}:S_p\to S_p\times\Sigma$$

dans ce cas aussi s'avèrent être pseudo-holomorphe. Notre but est de montrer ce qui suit.

Theorem 0.4.4. *Pour $x_1^1,\dots,x_p^1,x_1^2,\dots,x_p^2$ points fixes de φ, on a*

$$I_p((x_1^1\otimes\cdots\otimes x_p^1)\otimes(x_1^2\otimes\cdots\otimes x_p^2))\leq I(\mathcal{P}_p(x_1^1\otimes\cdots\otimes x_p^1)\otimes\mathcal{P}_p(x_1^2\otimes\cdots\otimes x_p^2))$$

Pour prouver ce théorème, nous sommes amenés à calculer des produits d'intersection à la Siefring $\bar{u}*\bar{v}$ entre les graphes des paires de pantalons u,v impliqués dans la définition de $\mathcal{P}_p$. Nous devons distinguer deux possibilités :

soit $u = v$, soit $u \neq v$. Le premier cas est très simple à traiter : un simple calcul de degré suffit, car I et I_p sont des fonctions des indices de Conley-Zehnder des points fixes $x_1^1, \ldots, x_p^1, x_1^2, \ldots, x_p^2$. Dans l'autre cas, il s'avère que $\bar{u} * \bar{v}$ borne par le bas la différence

$$I_p((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) - I(\mathcal{P}_p(x_1^1 \otimes \cdots \otimes x_p^1) \otimes \mathcal{P}_p(x_1^2 \otimes \cdots \otimes x_p^2))$$

de sorte qu'il suffit de prouver $\bar{u} * \bar{v} \geq 0$. La pseudo-holomorphie suffit à conclure cela, et prouve le théorème.

Chapter 1

Preliminaries

In this Chapter the reader will find a description of those notions that seem necessary to the understanding of this text. The Chapter is divided essentially in two parts: in the first we present braid groups for surfaces, and in the second one we talk about Hamiltonian diffeomorphisms, the geometry of Ham and the Floer homologies we use to approach their study.

1.1 Braid groups

Braid groups are classical objects in group theory. Typical references on the subject include the book [37] and the paper [28]. The name of the group was coined by Emil Artin in [6], where he gave an abstract definition in terms of generators and relations: for $k \geq 2$,

$$\mathcal{B}_k = \left\langle \sigma_1, \ldots, \sigma_{k-1} \middle| \begin{array}{c} |i-j| > 1 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i \\ 1 \leqslant i \leqslant k-2 \Rightarrow \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \tag{1.1}$$

There are more geometrical ways of seeing the braid groups, the one we are going to use in the present paper being the following one. The permutation group on k letters $\mathfrak{S}_k$ admits a faithful action on the complement in $\mathbb{D}^k$ of the set

$$\tilde{\Delta} = \left\{ (x_1, \ldots, x_k) \in \mathbb{D}^k | \exists i \neq j, x_i = x_j \right\}$$

and we call the quotient

$$\mathrm{Conf}^k(\mathbb{D}) := (\mathbb{D}^k \setminus \tilde{\Delta})/\mathfrak{S}_k$$

the configuration space of k unordered points of the disc. It turns out that $\pi_1(\mathrm{Conf}^{\,k}(\mathbb{D})) = \mathcal{B}_k$. This means essentially that after choosing k base points $p_1, \ldots, p_k$ on $\mathbb{D}$, one can define an element of $\mathcal{B}_k$ uniquely as the homotopy class of a path $\gamma : [0,1] \to \mathbb{D}^k \setminus \tilde{\Delta}$ such that there is $\sigma \in \mathfrak{S}_k$ verifying $\gamma(1) = \sigma\gamma(0)$, and that any braid may be realised this way. The multiplication in $\mathcal{B}_k$ then corresponds to concatenation of braids the obvious way.

There is a quotient homomorphism $\mathcal{B}_k \to \mathfrak{S}_k$, mapping positive and negative generators to the same transposition, or equivalently identifying σ_i with σ_i^{-1}. Its kernel is called P_k, the set of pure braids. One can visualise pure braids as braids such that, following the strands, bring each basepoint on itself.

In the following sections of this work we shall need the definition of linking number[1]

$$\mathrm{lk} : \mathcal{B}_k \to \mathbb{Z}$$

which is the only morphism of groups whose value on all the generators σ_i is 1. Remark that this convention, very natural from the algebraic viewpoint, requires a different normalisation from the standard one in topology: choosing basepoints 0 and $\frac{1}{2}$ in $\mathbb{D}$, the braid represented by $t \mapsto [0, \frac{1}{2}\exp(2\pi i t)]$ has linking number 2 according to our definition, while it is customarily used as example of loop with linking number 1 in geometric settings.

Another possible definition for the linking number, which only applies to pure braids, is the following (see [19] for a deeper explanation, including the case of capped braids). Consider a lift to $\mathbb{D}^k$ pure braid with k strands $b = [\gamma_1, \ldots, \gamma_k]$, be it $\tilde{b}$, and take any homotopy

$$h = (h_1, \ldots, h_k) : [0,1] \times [0,1] \to \mathbb{D}^k$$

starting at the trivial braid at the basepoints (each strand is constant there) and ending in $\tilde{b}$. One can prove that

$$\mathrm{lk}(b) = \sum_{i \neq j} (h_i \pitchfork h_j) \tag{1.2}$$

where on the right we count intersections with signs.

Remark 1.1.1. *Each transverse intersection in the count corresponds to a linking difference of 2, due to our normalisation* $\mathrm{lk}(\sigma_j) = 1$. *With the more usual convention general one considers the sum indexed on* $i < j$ *or divides the sum above by 2.*

By homotopy invariance of the intersection product, this quantity does not depend on the choice of (h_i); two different lifts of b being connected by a permutation of the strands, this definition does not depend on the choice of $\tilde{b}$ either. The linking number then quantifies how far a braid is from being trivial somehow, quantifying the failure of any homotopy h as above to be a homotopy between the trivial braid to $\tilde{b}$ through braids.

One may of course define the braid group for any compact, oriented surface with boundary[2] $\Sigma_{g,p}$. Geometrically speaking, the braid group $\mathcal{B}_{n,g,p}$ can be seen as the fundamental group of the n-configuration space of the surface $\Sigma_{g,p}$:

$$\mathcal{B}_{n,g,p} := \pi_1(\mathrm{Conf}^n(\Sigma_{g,p}))$$

[1] In the low-dimensional topology literature it may also be called “exponent”.

[2] The version for non orientable surfaces also exists, but it is not relevant here.

If $g \geq 1, p \geq 1$, the presentation in 1.1 gets substantially more complicated. We are going to report here the results we need from [10]. Let us fix n distinct base points on $\Sigma_{g,p}$. There exist then four families of generators (for a picture, see Figure 1.1):

- $\sigma_1, \ldots, \sigma_{n-1}$: they correspond to the generators of $\mathcal{B}_{n,0,1} \subset \mathcal{B}_{n,g,p}$, i.e. to half-twist swapping two of the base points, in such a way that the images of the paths are contained in a disc on the surface;
- $a_1, \ldots, a_g,\ b_1, \ldots, b_g$: they are obtained by homologically independent loops based at the first base point, P_1 (the other base points are instead fixed throughout the path). One way to describe them is seeing $\Sigma_{g,p}$ as the connected sum of g tori with p punctures, so that the loops a_i and b_i represent the generators of the homology of the i-th torus;
- $z_1, \ldots, z_{p-1}$: they correspond to loops based at P_1 and winding around the $i - th$ puncture exactly once (here as well the other base points are fixed). Remark that there are $p - 1$ generators, but p punctures. A loop around the last puncture may in fact be written as composition of the others.

The relations in this group are rather complex, and we are not going to report them here to keep the presentation lean. The interested reader will find a detailed account in [10, Theorem 1.1].

1.2 Symplectic manifolds and associated groups

We recall here some basic definition in Symplectic Topology. We do assume basic knowledge in differential geometry.

Definition 1.2.1 (Symplectic manifold)**.** *A smooth manifold M is said to be symplectic if there exists a de Rham 2-form ω which is closed and non degenerate. A symplectic manifold (M, ω) is exact if ω is exact.*

Basic examples of symplectic manifolds are $\mathbb{R}^2$ with the form $dx \wedge dy$, any cotangent bundle $\pi : T^*Q \to Q$ with its tautological symplectic form

$$\omega_{std} = d\lambda, \lambda_{(q,p)}(v) := p(d_{(q,p)}\pi.v)$$

and any compact, oriented surface $\Sigma_{g,p}$ of genus g and p boundary components, with its area form.

In the category of symplectic manifolds, the morphisms are diffeomorphisms that preserve the symplectic structure.

Definition 1.2.2 (Symplectomorphism)**.** *Let (M_0, ω_0), (M_1, ω_1) be a symplectic manifold. A diffeomorphism*

$$\varphi : (M_0, \omega_0) \to (M_1, \omega_1)$$

is said to be a symplectomorphism if $\varphi^*\omega_1 = \omega_0$.

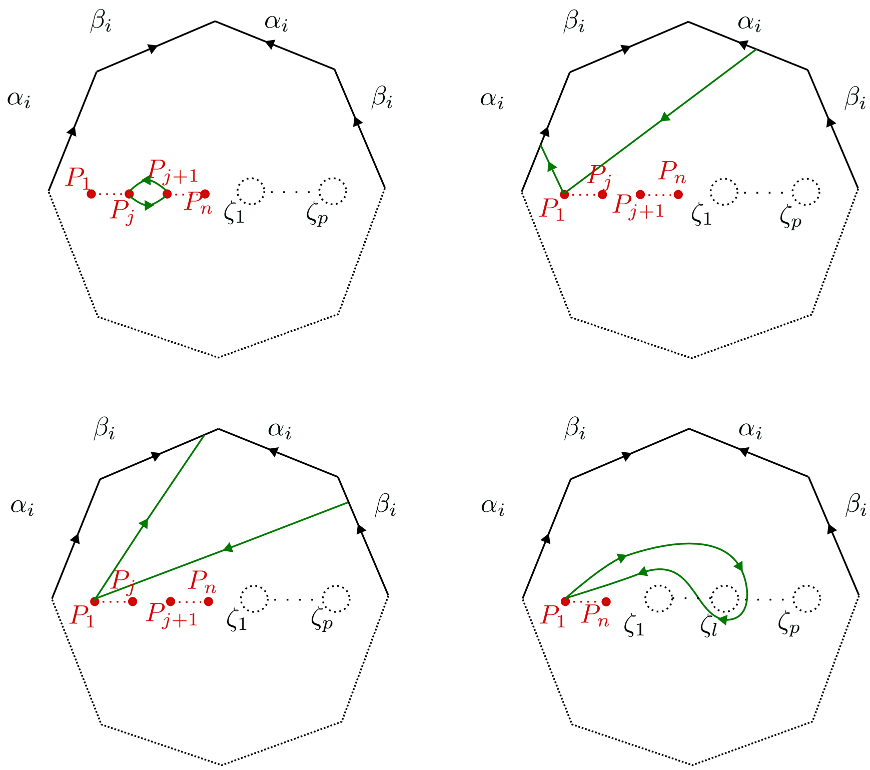


Figure 1.1: The generators of the braid group $\mathcal{B}_{n,g,p}$. In clockwise order, starting from top-left: σ_j, a_i, z_l and b_i . We draw a fundamental domain of the surface, and all base points which are not endpoints of the green paths are to be thought of as constant paths.

The symplectomorphisms of a symplectic manifold (M, ω) form a group, denoted by $\mathrm{Symp}(M, \omega)$. If we require the symplectomorphisms to be the identity outside a compact set of M, we talk about compactly-supported symplectomorphism. They also form a group, which we denote by $\mathrm{Symp}_c(M, \omega)$. We will drop the symplectic form whenever it is clear from context, or irrelevant.

A remarkable subgroup of $\mathrm{Symp}(M)$ is that of Hamiltonian diffeomorphisms. To introduce it, we have to define Hamiltonian vector fields first.

Definition 1.2.3. *Let (M, ω) be symplectic. Let $H \in \mathscr{C}^\infty(\mathbb{S}^1_t \times M; \mathbb{R})$. The Hamiltonian vector field associated to H, which we call X_H, is defined by the equation:*

$$\omega(X_{H_t}, \cdot) = -dH_t$$

The function H is called the Hamiltonian of X_H.

If the manifold M is not compact, we assume by default that all our Hamiltonians are null outside of a compact set: this makes the flow of X_H complete. We make a similar assumption if the manifold has boundary: we assume that the Hamiltonians are null outside of a compact set of the interior of the manifold (equivalently, they are null in a neighbourhood of the boundary).

Definition 1.2.4. *A diffeomorphism $\varphi \in \mathrm{Diff}(M)$ is Hamiltonian if there exists a function H as above such that φ is the time 1-map of X_H, denoted ϕ^1_H. The diffeomorphism φ is called autonomous if the Hamiltonian may be assumed to be time-independent.*

As anticipated above, every Hamiltonian diffeomorphism is symplectic. We denote by $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω). If M is non compact, our Hamiltonian diffeomorphisms will be compactly supported and the group denoted $\mathrm{Ham}_c(M, \omega)$. It will still be a normal subgroup of $\mathrm{Symp}_c(M, \omega)$. Given our conventions, we shall write $\mathrm{Ham}_c(M)$ throughout the present text, regardless of whether M is compact or not.

Given $\varphi \in \mathrm{Ham}_c(M)$, a Hamiltonian isotopy is a path in $\mathrm{Ham}_c(M)$ connecting the identity to φ. Any Hamiltonian H generating φ gives an isotopy to φ: the Hamiltonian flow

$$t \mapsto \phi^t_H$$

Remark 1.2.5. *It is implicit in the discussion above that our Hamiltonian isotopies are always compactly supported.*

There are several possible group norms on $\mathrm{Ham}_c(M)$. Here we are going to focus on the so-called Hofer norm. Given a compactly supported Hamiltonian H, we define its oscillation to be

$$\|H\| = \int_0^1 \left(\max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt$$

and the Hofer norm of a compactly supported Hamiltonian diffeomorphism by

$$\|\varphi\| = \inf_{H, \varphi = \phi^1_H} \|H\|$$

We define the Hofer distance between two Hamiltonian diffeomorphisms in a bi-invariant way:

$$d_H(\varphi, \psi) := \|\varphi\psi^{-1}\|$$

nondegeneracy of the Hofer distance is non trivial to show: it was proved in various degrees of generality in [30], [58], [42]. The Hofer norm retains non trivial dynamical information: see just as an example the results in [59] and [3]. A flourishing domain of study is also that of the large scale geometry of the Hofer norm: see for instance [20], [60] and references therein.

Remark 1.2.6. *It is in general complicated to find Hamiltonian diffeomorphisms with large Hofer norm. Known examples yield the following heuristics: in order to have large Hofer norm, a diffeomorphism should "move around big open sets in a complicated manner". The main result of Chapter 2 should be seen as a confirmation of these heuristics in a specific context.*

One of the main tools for the study of the Hofer metric on Ham are quasimorphisms, introduced in the field of Symplectic Topology in [22].

Definition 1.2.7. *A quasimorphism on a group G (in our case $G = Ham_c(\mathbb{D}, \omega)$) is a function $Q : G \to \mathbb{R}$ such that there is a constant $D \geq 0$, the defect of the quasimorphism, verifying for all $g, h \in G$*

$$|Q(gh) - Q(g) - Q(h)| \leq D$$

A quasimorphism is said to be homogeneous if it is a homomorphism when restricted to powers of the same element: $\forall g \in G, \forall k \in \mathbb{Z}$, $Q(g^k) = kQ(g)$. Given any arbitrary quasimorphism Q, we define its homogenisation by

$$\tilde{Q}(x) := \lim_{n\to\infty} \frac{Q(x^n)}{n}$$

Homogeneous quasimorphisms may be used to study properties of Hamiltonian diffeomorphism groups with respect to the Hofer metric when they are Hofer-Lipschitz. The research for and the study of Hofer-Lipschitz quasimorphisms has now a long history and a wide plethora of applications: some can for instance be found in [21] [22], [23], [38] [40], [61], and several others. The research for quasimorphisms on $\mathrm{Ham}_c(M, \omega)$ may be justified by the simplicity theorem of Banyaga [8] (see also [9] and [49]): for a closed manifold Ham is simple, so that it does not admit any non trivial homomorphism (kernels are normal subgroups). If however M^{2n} is an open manifold with symplectic form $\omega = d\lambda$ (for us $M = \mathbb{D}$ with its standard structure) there's a natural homomorphism on $\mathrm{Ham}_c(M, \omega)$:

$$\mathrm{Cal} : \mathrm{Ham}_c(M, \omega) \to \mathbb{D},\ \phi_H^1 \to \int_M H\omega^n$$

The morphism Cal will appear in the Appendix B as a part of definitions of new quasimorphisms.

We conclude our introduction to the group of Hamiltonian diffeomorphism with a classical result on surfaces. We write it down as a reference, since we could not find one elsewhere. It is contained in a joint work with Ibrahim Trifa [52], and it makes use of results of Gramain's, Moser's and Banyaga's which we are not going to introduce here as they are not required anywhere else.

Lemma 1.2.8. *If $\Sigma_{g,p} \neq \mathbb{S}^2$, $Ham_c(\Sigma_{g,p})$ is simply connected.*

Proof. Let $B = \partial\Sigma_{g,p}$: it is a disjoint union of p circles. The embedding $B \hookrightarrow \Sigma_{g,p}$ provides a fibration

$$\mathrm{Diff}_0(\Sigma_{g,p}, B) \hookrightarrow \mathrm{Diff}_0(\Sigma_{g,p}) \to \mathrm{Diff}_0(B) \tag{1.3}$$

where the fibre is the group of diffeomorphisms of $\Sigma_{g,p}$ inducing the identity on the boundary and isotopic to the identity, the total space is the connected component of the identity of the diffeomorphisms of $\Sigma_{g,p}$, and the base is the connected component of the identity in the group of diffeomorphisms of a disjoint union of circles.

Let us consider the long exact sequence of the fibration in (1.3):

$$\pi_2(\mathrm{Diff}_0(B)) \to \pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B)) \to \pi_1(\mathrm{Diff}_0(\Sigma_{g,p})) \to \pi_1(\mathrm{Diff}_0(B)) \tag{1.4}$$

The term $\pi_2(\mathrm{Diff}_0(B))$ is always 0, since it is a power of $\pi_2(\mathrm{Diff}_0(\mathbb{S}^1))$ and this is trivial ($\mathrm{Diff}_0(\mathbb{S}^1)$ has the homotopy type of the circle itself). This proves that the second arrow in the exact sequence (1.4) is always an injection. Now, if $g > 1$ or $g = 0, p \geq 3$, [29, Theorem 1] shows that $\mathrm{Diff}_0(\Sigma_{g,p})$ is contractible: in such a case we deduce that $\pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B)) = 0$. We want to prove that in this case $\mathrm{Diff}_0(\Sigma_{g,p}, B)$ and $\mathrm{Ham}_c(\Sigma_{g,p})$ have isomorphic fundamental groups.

Let $\mathrm{Symp}_0(\Sigma_{g,p}, B)$ be the group of symplectic diffeomorphisms of $\Sigma_{g,p}$ inducing the identity on the boundary and isotopic to the identity through symplectic diffeomorphisms with the same property. By [7] (generalising a result of Moser [53]) $\mathrm{Symp}_0(\Sigma_{g,p}, B)$ is a deformation retract of $\mathrm{Diff}_0(\Sigma_{g,p}, B)$, and in particular

$$\pi_1(\mathrm{Symp}_0(\Sigma_{g,p}, B)) \cong \pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B))$$

Fix a decreasing sequence of open collar neighbourhoods of the boundary, call them $(U_n)_{n\geq 1}$: they satisfy the property

$$\bigcap_{n\geq 1} U_n = B \tag{1.5}$$

Define by $\mathrm{Symp}_{U_n}(\Sigma_{g,p})$ the group of symplectic diffeomorphisms of $\Sigma_{g,p}$ which are supported in $\Sigma_{g,p} \setminus U_n$ and are isotopic to the identity via such diffeomorphisms.

Clearly,

$$\mathrm{Symp}_{U_n}(\Sigma_{g,p}) \subset \mathrm{Symp}_{U_{n+1}}(\Sigma_{g,p}), \quad \bigcup_{n\geq 1} \mathrm{Symp}_{U_n}(\Sigma_{g,p}) = \mathrm{Symp}_c(\Sigma_{g,p}) \tag{1.6}$$

and both these conditions together imply that

$$\pi_1(\mathrm{Symp}_c(\Sigma_{g,p})) = \lim_n \pi_1(\mathrm{Symp}_{U_n}(\Sigma_{g,p})) \tag{1.7}$$

We are now going to prove that $\pi_1(\mathrm{Symp}_c(\Sigma_{g,p}))$ injects into $\pi_1(\mathrm{Symp}_0(\Sigma_{g,p}, B)) = 0$ via the map induced by the inclusion. This will in turn imply that $\pi_1(\mathrm{Ham}_c(\Sigma_{g,p})) = 0$, since the map:

$$\pi_1(\mathrm{Ham}_c(\Sigma_{g,p})) \to \pi_1(\mathrm{Symp}_c(\Sigma_{g,p}))$$

induced by the inclusion is in fact an injection (see [49, Proposition 10.2.13]).

Let therefore $t \mapsto \varphi_t \in \mathrm{Symp}_c(\Sigma_{g,p})$ be a loop based at the identity which becomes contractible once seen as a representative in $\pi_1(\mathrm{Symp}_0(\Sigma_{g,p}, B))$. There is therefore a homotopy of symplectic isotopies fixing the boundary pointwise shrinking $t \mapsto \varphi_t$ to the constant path at the identity:

$$\psi : [0,1]_s \times [0,1]_t \to \mathrm{Symp}_0(\Sigma_{g,p}, B),$$
$$\psi(0,t) = \varphi_t, \psi(1,t) = \mathrm{Id}, \psi(s,0) = \psi(s,1) = \mathrm{Id}$$

The goal now is to modify this isotopy to a new one, in area-preserving diffeomorphisms which fix a small enough collar neighbourhood of the boundary. Since the support of φ_t is compact in $\Sigma_{g,p} \setminus B$ for all t, there exists an n big enough such that the support of φ_t is contained in $\Sigma_{g,p} \setminus U_n$ for all t.

Since we know that $\mathrm{Symp}_0(\Sigma_{g,p}, B)$ is a deformation retract of $\mathrm{Diff}_0(\Sigma_{g,p}, B)$, we may homotope ψ to another homotopy ψ', this time via diffeomorphisms fixing U_n pointwise (but not necessarily area-preserving), from the identity to itself. We now consider ψ' as a homotopy in diffeomorphisms of $\overline{\Sigma_{g,p} \setminus U_n}$: applying Moser's result again we find a homotopy in $\mathrm{Symp}_{U_n}(\Sigma_{g,p})$, call it ψ'', between the symplectic loop $t \mapsto \varphi_t$ and the constant loop. Since $\mathrm{Symp}_{U_n}(\Sigma_{g,p}) \subset \mathrm{Symp}_c(\Sigma_{g,p})$, the existence of ψ'' proves that we have a sequence of group morphisms

$$\pi_1(\mathrm{Ham}_c(\Sigma_{g,p})) \hookrightarrow \pi_1(\mathrm{Symp}_c(\Sigma_{g,p})) \hookrightarrow \pi_1(\mathrm{Symp}_0(\Sigma_{g,p}, B)) \cong \pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B))$$

and the rightmost group is 0 whenever $g > 1$ or $g = 0, p \geq 3$ by Gramain's result. We have thus proved that $\mathrm{Ham}_c(\Sigma_{g,p})$ is simply connected under the above topological assumptions.

We are now left with the cases of the disc $\mathbb{D}$ and cylinder Z to consider. In these two cases [29] shows that

$$\mathrm{Diff}_0(\mathbb{D}, \partial\mathbb{D}) \sim \mathrm{Diff}_0(Z, \partial Z) \sim SO(2, \mathbb{R})$$

where $\sim$ denotes homotopy equivalence, the diffeomorphisms induce the identity on the boundary, and $O(2, \mathbb{R})$ is the real orthogonal group of rank 2. Let us again adopt the notation from above: $\Sigma_{g,p}$ is our surface (disc or cylinder), and B its boundary. We have again an exact sequence

$$0 \to \pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B)) \to \pi_1(\mathrm{Diff}_0(\Sigma_{g,p})) \to \pi_1(\mathrm{Diff}_0(B))$$

but here the third group is not trivial. We need thus to show that the image of the second arrow is 0.

If $\Sigma_{g,p} = \mathbb{D}$, then $SO_2(\mathbb{R})$ is a subgroup of $\mathrm{Diff}_0(\Sigma_{g,p})$, and is in fact its deformation retract. A generator of $\pi_1(SO_2(\mathbb{R}))$ is given by a full rotation, which is mapped to the generator of $\pi_1(\mathrm{Diff}_0(\partial\mathbb{D}))$: the third arrow is an injection in the case of the disc, so that the map

$$\pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B)) \to \pi_1(\mathrm{Diff}_0(\Sigma_{g,p}))$$

from above is 0 as claimed.

If we examine $\Sigma_{0,2} = Z$ instead, the argument is similar: the generator of $\pi_1(\mathrm{Diff}_0(Z))$ is the full rotation, which is mapped to a nonzero element in $\pi_1(\mathrm{Diff}_0(\partial Z))$ (not a generator in this case, of course). The third arrow is an injection in this case as well.

Summing up, in the cases $g = 0, p \in \{1, 2\}$ we still have $\pi_1(\mathrm{Diff}_0(\Sigma_{g,p}, B)) = 0$. The rest of the arguments above did not depend on the actual surface we worked on, and they carry over to this context: we infer that

$$\pi_1(\mathrm{Ham}_c(\mathbb{D})) \cong \pi_1(\mathrm{Ham}_c(Z)) = 0$$

□

Almost complex structures Symplectic manifolds always admit almost complex structures. They are going to be used in the following sections to describe the construction of Floer theories.

Definition 1.2.9. *Given a manifold M, an almost complex structure J on M is an endomorphism of its tangent bundle such that $J^2 = -\mathrm{Id}$.*

The standard complex structure on $\mathbb{C}$

$$z \mapsto \sqrt{-1}z$$

is the most basic example of almost complex structure.

The set of almost complex structure is contractible, like the set of tame and compatible almost complex structures, whose definitions now follow.

Definition 1.2.10. *Let (M, ω) be a symplectic manifold. An almost complex structure J is said to be ω-tame if*

$$\forall x \in M, \forall v \in T_xM \setminus \{0\}, \omega(v, Jv) > 0$$

The almost complex structure J is said to be ω-compatible if furthermore J is a symplectic transformation:

$$\omega(J\cdot, J\cdot) = \omega$$

Lagrangian submanifolds Lagrangian submanifolds are maximal isotropic manifolds, and they exhibit strong rigidity properties when deformed by Hamiltonian diffeomorphisms.

Definition 1.2.11. *A submanifold $(L, \iota : L \to M)$ of a symplectic manifold (M, ω) is said to be Lagrangian if*

$$\iota^*\omega = 0 \text{ and } 2 \dim L = \dim M$$

A Lagrangian (L, ι) in an exact symplectic manifold $(M, \omega = d\lambda)$ is said to be exact if there exists

$$f_L \in \mathscr{C}^\infty(L; \mathbb{R}) \text{ with } df_L = \iota^*\lambda$$

Simple examples of exact Lagrangians are graphs of differentials in cotangent bundles.

1.3 Hamiltonian Dynamics and Variational Principles

In this section we are going to describe the theories we apply in the text. We start by giving some elementary definitions, proceed then with a paragraph more historical in nature, which we use to introduce the notion of Hamiltonian twist maps on surfaces. With this in our hands, we describe the classical theory of generating functions in cotangent bundle, which may be used to describe Hamiltonian diffeomorphisms. After that we move to the Floer side: we sketch the construction of Quantitative Heegaard-Floer homology and the associated spectral invariants following [21], and to finish we talk about Hamiltonian Floer Homology, its product operations and the Siefring product for punctures holomorphic curves in symplectic cobordisms.

Let φ be a Hamiltonian diffeomorphism of a symplectic manifold (M, ω).

Definition 1.3.1. *A point $x \in M$ is a fixed point of φ if $\varphi(x) = x$. In such a case, we write $x \in \mathrm{Fix}(\varphi)$. A fixed point is contractible if, for any choice of path in $Ham_c(M)$ between the identity and φ, the loop $t \mapsto \varphi_t(x)$ is contractible. This notion does not depend on the isotopy one considers.*

An iterated of φ is simply a power φ^k, for an integer $k > 0$. By definition, $\varphi^0 = \mathrm{Id}_M$. There is a composition formula for Hamiltonian functions: if H and K are Hamiltonians on M generating respectively φ and ψ, the function

$$H\#K(t, x) := H(t, x) + K(t, (\varphi^t)^{-1}x)$$

generates $\varphi \circ \psi$. In fact, more is true: it generates the isotopy

$$t \mapsto \phi^t_H \circ \phi^t_K$$

Definition 1.3.2. *A periodic point x of φ is a fixed point of an iterate. The period k of x is*

$$\min \left\{ k \in \mathbb{N}_{>0} | x \in \mathrm{Fix}(\varphi^k) \right\}$$

For the next definition, remark that given $x \in \mathrm{Fix}(\varphi)$, $d_x\varphi$ is an endomorphism of T_xM.

Definition 1.3.3. *A fixed point x is non degenerate if 1 is not an eigenvalue of $d_x\varphi$. A Hamiltonian diffeomorphism is non degenerate if every $x \in \mathrm{Fix}(\varphi)$ is non degenerate.*

It is possible to show that non degenerate critical points are isolated.

Remark 1.3.4. *Our Hamiltonian diffeomorphisms of $\mathbb{R}^2$ will always be compactly supported, and as such cannot be non degenerate in the above sense. When we say that $\varphi \in Ham_c(\mathbb{R}^2)$ is non degenerate, we mean that all critical points in the interior of the support of φ are non degenerate. If our goal is to define a Floer theory for φ, we perturb a generating Hamiltonian by a summand which outside of a compact set is a small affine function of the radius. When necessary then we tacitly assume to fix a perturbation of a compactly supported Hamiltonian diffeomorphism.*

Without giving a proper definition for it, we introduce the Conley-Zehnder index. Let $x \in \mathrm{Fix}(\varphi)$ be non degenerate: the integer denoted by $CZ(x)$ is its Conley-Zehnder index. It essentially quantifies how the infinitesimal flow twists a symplectic frame of reference at the fixed point during a Hamiltonian isotopy. Accurate definitions may for instance be found in [18], but see [63] for a more general case. The definition of the index however already appears in [27].

1.3.1 Poincaré-Birkhoff theorem and Arnol'd conjecture

The content of this section is classical: a good introductory reference may be found for instance in [65]. Before stating the famous Poincaré-Birkhoff Theorem, we have to introduce a remarkable class of diffeomorphisms of the plane:

Definition 1.3.5. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving diffeomorphism. We say that ϕ is a twist map if, denoting $\phi(x_0, y_0) = (x_1, y_1)$, we have the inequality*

$$\frac{\partial x_1}{\partial y_0} > 0$$

Geometrically, this may be visualised as follows: given any vertical line of constant x coordinate, it is sent to the graph of a strictly increasing function. The definition may be extended to diffeomorphisms realising the inequality $\frac{\partial x_1}{\partial y_0} < 0$, in which case vertical lines are mapped to graphs of strictly decreasing functions.

Remark 1.3.6. *This interpretation justifies the French denomination of twist maps: they are called applications déviant la verticale, literally "functions deviating the vertical". The vertical will be deviated à droite ("to the right") or à gauche ("to the left") in respectively the first and second case.*

Example. *The prototypical example of a twist map is the Dehn twist*

$$D : (x, y) \mapsto (x + y, y)$$

It is apparent that such a diffeomorphism is a twist map (and in fact it is déviant la verticale à droite). The clockwise rotation

$$R : (x, y) \mapsto (y, -x)$$

is also a twist map, again à droite.

Let us now moreover assume that the twist map ϕ is symplectic. In such a case, there exists what is called a generating function for ϕ: it is a map $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\forall (x, y), (x', y') \in \mathbb{R}^2,\ \phi(x, y) = (x', y') \Leftrightarrow \begin{cases} y = -\partial_1 h(x, x') \\ y' = \partial_2 h(x, x') \end{cases} \tag{1.8}$$

Example. *An example of generating function for the Dehn twist above is*

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \ (x, x') \mapsto \frac{1}{2}(x' - x)^2 \tag{1.9}$$

while the clockwise rotation is generated by

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \ (x, x') \mapsto -xx' \tag{1.10}$$

Generating functions make their entrance in the picture in the proof of Poincaré-Birkhoff Theorem. Before stating it, we extend the definition of twist map to the annulus: $\phi : \mathbb{S}^1 \times (a, b) \to \mathbb{S}^1 \times (a, b)$ is a twist map if on top of the conditions above (being orientation preserving and satisfying the inequality on the derivatives) it preserves the two boundary components of the annulus, and is moreover a rotation there.

On an annulus to a periodic point x of ϕ of period $p \geq 1$ one may associate a rotation number $\frac{q}{p}$. The number $q \in \mathbb{Z}$ is the degree of the composition $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t)) \mapsto \gamma_1(t)$. In this definition, $\gamma(t)$ is a Hamiltonian path in $\mathbb{S}^1 \times (a, b)$ of length p for ϕ such that $\gamma(0) = \gamma(p) = x$, and γ_i are its projections onto the two factors of the annulus.

Poincaré-Birkhoff Theorem, in its original statement, therefore reads:

Theorem 1.3.7 (Poincaré-Birkhoff)**.** *Let ϕ be a twist map of the annulus, and $\theta_1, \theta_2 \in \mathbb{R}$ be the rotation angles of the two boundary components. Assume that $\theta_1 < 0, \theta_2 > 0$. Then, if $\frac{q}{p} \in (\theta_1, \theta_2)$ is rational, there exists a pair of p-periodic points of ϕ with rotation number $\frac{q}{p}$.*

The proof is in fact given by an application of Weierstrass theorem. We first consider a lift of the problem to the universal cover of the annulus, $\mathbb{R} \times (a, b)$: a

$\frac{q}{p}$-periodic point of ϕ is then a point $x = (x_1, x_2)$ such that $\phi^p(x) = (x_1 + q, x_2)$. We then apply Weierstrass theorem to the function

$$H(\xi_0, \dots, \xi_{p-1}) = \sum_{i=1}^{p-1} h(\xi_i, \xi_{i+1})$$

defined on the compact subspace of $\mathbb{R}^{2p}$ given by the constraints

$$\xi_0 \leq \xi_1 \leq \cdots \leq \xi_{p-1} \leq \xi_p = x_0 + q$$

One then has to verify that the global maximum and the global minimum do not lie on the boundary (this would give rise to a “degenerate orbit”, i.e. an orbit of lower period). The projection on $\mathbb{S}^1$ of the first coordinate ξ_0 of the maximum or the minimum is the first coordinate of a $\frac{q}{p}$-periodic orbit of ϕ.

Historically, the proof of Poincaré-Birkhoff Theorem and the existence together with the existence of a second kind of generating functions (explained in the next section) motivated Arnol'd conjecture, one of the main reasons for the very existence of the field of Symplectic Topology:

Conjecture 1.3.8 (Arnol'd Conjecture - Hamiltonian version)**.** *Let ϕ be a compactly-supported, non degenerate Hamiltonian diffeomorphism of a symplectic manifold M^{2n}, and let b_i be the i-th integral Betti number of M. Then*

$$\#(\mathrm{Fix}(\phi)) \geq \sum_{i=0}^{2n} b_i$$

There also exists a more general Lagrangian formulation:

Conjecture 1.3.9 (Arnol'd Conjecture - Lagrangian version)**.** *Let M^{2n} be a compact symplectic manifold, and L and embedded Lagrangian submanifold. If a Hamiltonian diffeomorphism ϕ is such that $\phi(L) \pitchfork L$, and we denote by b_i be the i-th integral Betti number of L. Then*

$$\#(\phi(L) \cap L) \geq \sum_{i=0}^{n} b_i$$

The Lagrangian Arnol'd conjecture implies the Hamiltonian one in the compact setting: if $\phi \in \mathrm{Ham}_c(M, \omega)$, one may to consider the intersections between diagonal Δ (Lagrangian in $(M, \omega) \oplus (M, -\omega)$) and the graph of ϕ.

The theories explained in the following sections prove Arnol'd conjecture under some geometrical hypotheses on the involved manifolds.

1.3.2 Generating functions à la Viterbo

The first construction we would like to introduce is the one of generating functions, used to study Lagrangian intersections in cotangent bundles.

In the following, M is going to be a closed manifold or $\mathbb{R}^{2n}$. Endow T^*M with its canonical exact symplectic structure.

Definition 1.3.10 (Generating functions)**.** *Let $L \subset T^*M$ be an exact Lagrangian submanifold. We say that $S : E \to \mathbb{R}$ is a generating function for L if:*

i) *$\pi : E \to M$ is a real vector bundle on M;*

ii) *S is vertically transverse to 0. By this we mean that we define the vertical differential of S, let us denote it by*

$$\partial^V S : E \to (\ker d\pi)^*$$

and we ask that $\partial^V S \pitchfork 0$.

iii) *L is the image of the following Lagrangian immersion. Let $\Lambda_S := \partial^V S^{-1}(0)$ (it is a submanifold of E by previous point), and define the Lagrangian immersion by*

$$\iota_S : \Lambda_S \to T^*M, \;\; (x, \xi) \mapsto (x, \partial^H S(x, \xi))$$

In the third point, $\partial^H S(x, \xi) \in T^*M$ is defined as

$$(x, v) \mapsto d_{(x,\xi)} S.v', \; \textit{for any } v' \in T_{(x,\xi)}E, d_{x,\xi}\pi.v' = v$$

By definition, the critical set of a generating function is in bijection with the intersection points between such Lagrangian and the zero section of T^*M, so if we want to bound the cardinality of the latter we are naturally lead to do Morse theory on a generating function. A vector bundle is however never a compact manifold, so we need to work within a class of generating functions which satisfy the Palais-Smale compactness condition.

Definition 1.3.11. *A generating function $S : E \to \mathbb{R}$ is said to be quadratic at infinity (and we shall write GFQI for it) if there exists a function $Q : E \to \mathbb{R}$ which when restricted to the fibres is a non degenerate quadratic form, and such that $S - Q$ is supported on a compact set.*

Remark 1.3.12. *This definition may in fact be relaxed a bit (see [69]), but we will not need the extended definition.*

Given a GFQI, S, we denote by $\sigma(S)$ its signature, defined to be the signature of the quadratic form at infinity (maximal dimension of a subspace on which it is negative definite).

To answer our intersection-counting question via Morse theory, we need an existence statement: as of now, we are not sure for what class of Lagrangian submanifolds one can construct generating functions, if they ever exist at all.

Theorem 1.3.13 (Sikorav '87 [68])**.** *Let us assume that $L \subset T^*M$ admits a generating function: then if $\varphi \in Ham(T^*M)$, so does $\varphi(L)$.*

Remark 1.3.14. *It is in fact possible to prove (see [69]) that the map from generating functions to Lagrangian submanifolds is an infinite dimensional smooth Serre fibration in a precise sense.*

The set of Lagrangians which admit generating functions is therefore closed under Hamiltonian isotopies. Since the zero section $0_M \subset T^*M$ is generated by any non degenerate (fibre-independent) quadratic form, at the very least any Hamiltonian deformation of the zero section admits a generating function.

For general knowledge, we choose to now state the Nearby Lagrangian Conjecture:

Conjecture 1.3.15. *Any closed exact Lagrangian in T^*M is Hamiltonian isotopic to the zero section.*

Remark 1.3.16. *This conjecture is not a Theorem yet.*

Existence of a generating function alone however does not guarantee the possibility of counting intersections by means of Morse theory, as mentioned above. Moreover, it is not clear how different generating functions for the same Lagrangian submanifold are related to one another. We introduce then certain elementary operations using which, given a generating function, we may obtain new ones for the same Lagrangian submanifold.

Definition 1.3.17. *Assume $S : E \to \mathbb{R}$ is a generating function on $\pi : E \to M$ for the Lagrangian $L \subset T^*M$ We call the following operations elementary:*

(Shift) *If $c \in \mathbb{R}$, $S + c$ is a new generating function for L, defined again on E;*

(Gauge equivalence) *If ϕ is a vector bundle automorphism of E,*

$$S \circ \phi : E \to \mathbb{R}$$

is a new generating function for L, defined on the bundle

$$\pi \circ \phi : E \to M$$

(Stabilisation) *If $Q : E' \to M$ is a non degenerate quadratic form on the vector bundle $\pi' : E' \to M$, then*

$$S + Q : E \oplus E' \to \mathbb{R}$$

is a generating function for L, defined on

$$\pi \oplus \pi' : E \oplus E' \to M$$

Remark 1.3.18. *We may, without loss of generality, always assume to be adding trivial bundles for the (Stabilisation) operation. In fact, if $E' \to M$ is a vector bundle, there exists another vector bundle on M, say E'', such that $E' \oplus E''$ is trivial. Moreover, it is remarked in [69] that any non degenerate quadratic form is equivalent diffeomorphic by a gauge equivalence to one which does not depend on the point of the base, i.e. if the vector bundle E is globally trivial, and $p, q \in B$, then $Q|_{\pi^{-1}(p)} = Q|_{\pi^{-1}(q)}$.*

We may now state Viterbo's Uniqueness Theorem:

Theorem 1.3.19 (Viterbo, [73])**.** *A Lagrangian submanifold $L \subset T^*M$ is said to have the GFQI-uniqueness property if, given $S : E \to \mathbb{R}$, $S' : E' \to \mathbb{R}$ generating functions for L, there exist two finite sequences of elementary operations taking both S and S' to the same $S'' : M \times \mathbb{R}^N \to \mathbb{R}$, GFQI for L. Then:*

i) If L has the GFQI-uniqueness property, and $\varphi \in Ham(T^*M)$, then so does $\varphi(L)$;

ii) The zero section 0_M has the GFQI-uniqueness property.

Proof. For a complete proof of the Theorem, see [69]. □

Remark 1.3.20. *If $L \subset T^*M$ is a Lagrangian with admitting a generating function (its quadraticity at infinity is not necessary) S, S is Morse if and only if $L \pitchfork 0_M$.*

Lagrangian Arnol'd conjecture is now proved in the context of Hamiltonian deformations of the zero section of cotangent bundles.

Generating functions may be used to describe compactly supported Hamiltonian diffeomorphisms of $(\mathbb{R}^{2n}, \omega = \sum_i dx^i \wedge dy^i)$: denoting by $\overline{\mathbb{R}^{2n}}$ the symplectic manifold $(\mathbb{R}^{2n}, -\omega)$, if Δ is the diagonal of $\overline{\mathbb{R}^{2n}} \oplus \mathbb{R}^{2n}$, there is a symplectic identification

$$\overline{\mathbb{R}^{2n}} \oplus \mathbb{R}^{2n} \to T^*\Delta$$

mapping Δ to the zero section, given by

$$(x, y, X, Y) \mapsto (x, Y, y - Y, X - x) \tag{1.11}$$

We have endowed above $T^*\Delta$ with its tautological exact symplectic form. Given a Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$ with compact support, we push forward its graph to $T^*\Delta$ by the symplectic identification above. One may then find a generating function for the Lagrangian deformation of the zero section in $T^*\Delta$ one obtains this way. Fixed points of the Hamiltonian diffeomorphism are in bijection with the Lagrangian intersections between the zero section and its deformation, and thus with the critical points of the generating function. Let $\varphi \in \mathrm{Ham}_c(\mathbb{R}^{2n})$, and

$$h : \Delta \times \mathbb{R}^N_\xi \to \mathbb{R}$$

a generating function of its graph in $T^*\Delta$. The map in (1.11) yields then the equivalence

$$\varphi(x, y) = (X, Y) \Leftrightarrow \begin{cases} \partial_\xi h(X, y; \xi) = 0 \\ X - x = \partial_Y h(X, y; \xi) \\ Y - y = -\partial_x h(X, y; \xi) \end{cases} \tag{1.12}$$

Remark 1.3.21. *Let us remark that in principle the kind of generating functions we have just talked about is a different one from the ones earlier in this Chapter, for twist maps.*

Remark 1.3.22. *A diffeomorphism $\varphi \in Ham_c(\mathbb{R}^{2n})$ is non degenerate if and only if for any GFQI S with quadratic form Q that represents it, $S-Q$ is Morse in the interior of its support.*

We now quote a Lemma as reported by Brunella about composition formulas for GFQI.

Lemma 1.3.23 ([14])**.** *Let $L \subset T^*\mathbb{R}^{2n}$ be an immersed Lagrangian submanifold with a GFQI $S : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ coinciding with $Q : \mathbb{R}^k \to \mathbb{R}$ at infinity. Let g be a compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{4n}$ with generating function $F : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$. Then $g(L)$ has a GFQI with fibres of dimension $4n + k$.*

This Lemma is proved providing an explicit formula.

Morse theory for GFQI on $\mathbb{R}^{2n}$ Trying to define Morse theory for a GFQI $S : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ of a non degenerate, compactly supported Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}_c(\mathbb{R}^{2n})$, one immediately runs into the problem that S is never going to be Morse on the whole of $\mathbb{R}^{2n}$. It is possible to try to define the whole Morse complex as a limit of complexes of perturbations in a given family. We are not going to proceed this way, as considering the limit one has to take the homology first for the continuation maps to be well defined, thus losing the correspondence generator$\leftrightarrow$fixed point.

An alternative approach is to fix a single perturbation of S, assuming it has only finitely many critical points: this is possible for instance making the hypothesis that the perturbation is radial with respect to the $\mathbb{R}^{2n}$-coordinate at infinity, without critical points outside a compact set. It is possible to read this text supposing that all functions be really Morse, and the pairs function-metric to be both Morse-Smale and Palais-Smale.

We now present an approach that has the advantage of retaining perturbation-independent information only. It has been inspired by an analogous constrtuction in [3].

Definition 1.3.24. *Let $\varphi \in Ham_c(\mathbb{R}^{2n})$, and $x \in \mathrm{Fix}(\varphi)$. For a fixed generating function S of φ, the S-action of x is defined to be the critical value of the critical point of S corresponding to x. The set of all critical values of S is called the spectrum of S, and will be denoted by* $\mathrm{Spec}(S)$.

Remark 1.3.25. *For a fixed $\varphi \in Ham_c(\mathbb{R}^{2n})$, the spectrum depends on the generating function we choose up to translation, because of the (Shift) operation. If we restrict our attention to GFQIs however the spectrum is well defined.*

Assume that φ has no non degenerate fixed points of S-action 0. This condition is generic in $\mathrm{Ham}_c(\mathbb{R}^{2n})$, as it may be seen using the definition of action given in Section 1.3.4, which is equivalent to the one above. Perturb the generating function S as above (we call the perturbation S again). Fix any $a < b$ with $0 \notin (a, b)$, or equivalently $ab > 0$, allowing $a = -\infty$ or $b = +\infty$ when possible, and any g making (S, g) both Palais-Smale and Morse-Smale. One can then define

$$CM^a(S, g; \mathbb{Z}) := \bigoplus_{x \in \mathrm{Crit}(S), S(x) < a} \mathbb{Z} \cdot x$$

and similarly $CM^b(S, g; \mathbb{Z})$.

Remark 1.3.26. *Assume the perturbation of S to be small enough. If $a > 0$ both subcomplexes contain the non degenerate critical points obtained from perturbing the degenerate critical points of 0 S-action. If $b < 0$, such points do not appear in either subcomplex.*

We then define the Morse complex in the action window (a, b) to be

$$CM^{(a,b)}(S, g; \mathbb{Z}) := CM^b(S, g; \mathbb{Z})/CM^a(S, g; \mathbb{Z}) \tag{1.13}$$

where the differential counts gradient lines connecting generators. In the text, whenever we write

$$CM(S, g; \mathbb{Z})$$

for a generating function S of a given compactly supported Hamiltonian diffeomorphism, we mean the collection

$$\left(CM^{(a,b)}(S, g; \mathbb{Z})\right)_{a<b, 0 \notin (a,b)} \tag{1.14}$$

as defined in (1.13).

We now shift the grading of the complex much like in the work of Traynor [70] (this operation does not affect what has been done above). Let $\sigma(S)$ the signature of a GFQI S. If $x \in \mathrm{Crit}(S)$ has Morse index k, it is a generator of $CM(S, g; \mathbb{Z})$ in degree $k - \sigma(S)$. With this convention, the three elementary operations of Definition 1.3.17 induce isomorphisms of graded complexes. The grading shift is clearly needed due to the stabilisation operation, which in general changes the Morse index.

Continuation maps We now sketch the construction for Morse continuation maps. Let S_0, S_1 be GFQI $\mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ for two compactly supported non degenerate Hamiltonian diffeomorphisms. Let Q_0 and Q_1 be the two asymptotic non degenerate forms, which we may assume to be independent of the base point $x \in \mathbb{R}^{2n}$. We require that the S_i be defined on the same space (we did so above implicitly), and that $Q_0 = Q_1$: up to stabilisation they have the same signature, and then apply a fibre-preserving diffeomorphism taking an orthogonal basis of one to an orthogonal basis of the other. As highlighted above, S_0 and S_1 will be non degenerate only in the interior of the supports of $S_i - Q_i$. We ignore this problem in the exposition, using the above method. Fix Riemannian metrics g_i such that, up to small perturbations, the (S_i, g_i) are Morse-Smale and Palais-Smale pairs.

Define the function

$$S : [0, 1] \times \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}, \qquad S(t, x) := (1 - t)S_0(x) + tS_1(x)$$

and the metric on $[0,1]\times\mathbb{R}^{2n}\times\mathbb{R}^k$

$$g(t,x) := (1-t)g_0(x) + tg_1(x) + dt^2$$

Perturb S so that it has a smooth extension to $(-\delta, 1+\delta)\times\mathbb{R}^{2n+k}$, for a small $\delta>0$. Likewise, extend g on the same set. Define a smooth $h:(-\delta,1+\delta)\to\mathbb{R}$ such that for a $\varepsilon>0$ we have

- $h(0) = \|S_0 - S_1\|_\infty + \varepsilon$
- $h(1) = 0$
- $h'(t) < 0$ for all $t\in(0,1)$
- 0 is a point of global maximum for h, and 1 one of global minimum.

Remark that such function h may be constructed for an arbitrarily small value of ε. Then we also may assume that

$$\forall t\in(-\delta,1+\delta), \qquad |h'(t)| > \|S_1 - S_0\|_\infty \tag{1.15}$$

Because of this inequality, studying the Morse complex of

$$(t,x)\mapsto h(t) + S(t,x)$$

together with the perturbation of the metric g above, one may define a map

$$CM(S_0,g_0;\mathbb{Z})\to CM(S_1,g_1;\mathbb{Z})$$

In fact, (1.15) forces negative gradient lines to connect critical points of S with $t=0$, which are in bijection with critical points of S_0, to critical points of S with $t=1$, which are secretely critical points of S_1.

The complexes above are defined if the S_i are Morse. Otherwise, we are forced to look at the homology of the complexes $CM^{(a,b)}(S_i,g_i;\mathbb{Z})$. To describe continuation maps in this case, we need to be able to describe how the value of the filtration changes.

Lemma 1.3.27. *Let $x_0\in \mathrm{Crit}(S_0)$ be a generator of the Morse complex for (S_0,g_0). Assume $x_1\in\mathrm{Crit}(S_1)$ is a generator for the target Morse complex such that there exists a negative gradient line for $(h+S,g)$*

$$\gamma:\mathbb{R}\to(-\delta,1+\delta)\times\mathbb{R}^{2n+k},\ \ \dot\gamma(\sigma) = -\nabla^g(h+S)(\gamma(\sigma))$$

negatively asymptotic to $(0,x_0)$ and positively asymptotic to $(1,x_1)$

$$\lim_{\sigma\to-\infty}\gamma(\sigma) = (0,x_0),\ \ \lim_{\sigma\to+\infty}\gamma(\sigma) = (1,x_1)$$

Then

$$S_1(x_1) - S_0(x_0) \le \|S_0 - S_1\|_\infty + \varepsilon \tag{1.16}$$

where ε appears in the definition of h.

Proof. Because γ is a negative gradient line for $h+S$, clearly

$$(h+S)(1,x_1) = (h+S)(\gamma(+\infty)) \leq (h+S)(\gamma(-\infty)) = (h+S)(0,x_0)$$

By definition therefore

$$S_1(x_1) - S_0(x_0) \leq h(0) - h(1) = \|S_0 - S_1\|_\infty + \varepsilon$$

□

One inconvenient of the definition in (1.14) is that continuation maps get more complicated to describe, since the chain complex is not defined for all action values at once. The Lemma above lets us define the continuation maps for fixed intervals of values of the Morse functions.

Corollary 1.3.28. *Let $a < b$ with $ab > 0$, and write*

$$\lambda := \|S_0 - S_1\|_\infty$$

Then, for $\varepsilon > 0$ small enough, the following continuation maps are defined if (and only if) both $ab > 0, (a+\lambda)(b+\lambda) > 0$:

$$CM^{(a,b)}(S_0, g_0; \mathbb{Z}) \to CM^{(a+\lambda+\varepsilon, b+\lambda+\varepsilon)}(S_1, g_1; \mathbb{Z})$$

Proof. The action shift is clear by the above Lemma. We only need to be mindful about the definition of the Morse complex of $h+S$: we have to show that as soon as the Morse complexes for the S_i above are defined, then so is the one of $h+S$. To define the morphism we use the differential of the Morse complex

$$CM^{(a+\lambda+\varepsilon, b+\lambda+\varepsilon)}(h+S, g; \mathbb{Z})$$

For ε small enough there are no degenerate critical points of $h+S$ in this interval. Critical points in $\{0\} \times \mathbb{R}^{2n+k}$ coincide with critical values of S_0 by construction, but the critical value is increased by λ, so the morphism is indeed defined where claimed. □

In particular this Corollary shows that continuation maps in this framework may therefore be defined for small $\mathscr{C}^0$-deformations of the generating function only, the size of which depends on the actual action window (a,b) we start from.

1.3.3 Quantitative Heegaard-Floer Homology

This section is the most technical of the introduction. It will set the notation and the groundwork for Chapter 2. We use here the conventions from [47].

The tool we shall use is the family of quasimorphisms defined in [21]: we are going now to sketch their construction, and refer to that paper for details. These quasimorphisms come as homogenisation of spectral invariants of a Lagrangian Floer theory associated to a collection of curves on a surface.

To fix notation, if Σ is a closed surface, $\mathrm{Sym}^{k+g}(\Sigma) := \Sigma^k/\mathfrak{S}_k$ will denote its k-fold symmetric product, and Δ will be its singular locus, i.e. the fat diagonal

$$\Delta = \{\ (z_1, \dots, z_k) | \exists i \neq j, z_i = z_j\ \} / \mathfrak{S}_k$$

The action of $\mathfrak{S}_k$ is given by permutation of the factors: for $(z_1, \dots, z_k) \in \Sigma^k$ and $\sigma \in \mathfrak{S}_k$,

$$\sigma \cdot (z_1, \dots, z_k) := (z_{\sigma(1)}, \dots, z_{\sigma(k)})$$

If Σ is also oriented, it is symplectic and the symplectic form $\omega^{\oplus k}$ descends to a singular symplectic form $\omega_{\mathrm{Sym}^{k+g}(\Sigma)}$ on the symmetric product. While the symmetric product of manifolds in full generality is not a manifold but an orbifold, one can give a structure of complex manifold to symmetric products of Riemann surfaces. The symplectic structure on the quotient can be smoothed out using a procedure devised by Perutz following a paper by Varouchas: for details we refer to [57] and [21]. The fact that we use a Perutz-type symplectic form on the symmetric product will always be in the background without being explicitly mentioned here, but it is fundamental in the construction of the homology theory we are about to describe carried out in [21].

In what follows, $X := \mathrm{Sym}^{k+g}(\Sigma_g)$, and ω_X is the singular symplectic form induced by the symplectic form on Σ_g.

Let $L_1 \times \dots \times L_{k+g} =: \underline{L} \subset X$ be a collection of k non intersecting circles on Σ_g (i.e. $\underline{L} \cap \Delta = \emptyset$). We assume $L_1, \dots, L_k$ to be contractible. Let (B_j) be the collection of connected components of $\Sigma_g \setminus \bigcup_{i=1}^{k+g} L_i$, k_j be the number of connected components of ∂B_j, and $A_j = Area(B_j)$. The collection $\underline{L}$ is said to be η-**monotone**, for some $\eta \geq 0$, if there exists a $\lambda > 0$ such that for every j

$$\lambda = 2\eta(k_j - 1) + A_j \tag{1.17}$$

The circles in $\underline{L}$ will be required to satisfy some topological conditions as well: in [21] it is required that the cloures of the B_j in Σ be of genus 0, but as we show in Section 2.3.1 we may relax this assumption. In this text we shall assume that the g non contractible circles in $\underline{L}$ be homologically independent meridians in Σ_g.

Remark that $\mathrm{Sym}^{k+g}(\underline{L})$ is in fact a Lagrangian in the symmetric product, and the authors of [21] proceed to the definition of a Lagrangian Floer complex associated to any Hamiltonian deformation of it. More precisely, let $H : \Sigma_g \times \mathbb{S}^1 \to \mathbb{R}$ be a periodic Hamiltonian of time 1 map $\varphi = \phi_H^1$: it naturally defines a Hamiltonian function $\mathrm{Sym}^{k+g}(H)$ and a (non smooth) Hamiltonian diffeomorphism of the symmetric product which we denote $\mathrm{Sym}^{k+g}(\varphi)$. Assume that for all $i \neq j \in \{1, \dots, k\}$ we have $\varphi(L_i) \pitchfork L_j$, and let $\mathbf{x} \in \mathrm{Sym}^{k+g}(\underline{L})$. Let us consider $\tilde{\mathcal{S}}$ the set of all Hamiltonian paths from $\mathrm{Sym}^{k+g}(\underline{L})$ to itself which are homotopic to the constant path $\mathbf{x}$ through paths between $\mathrm{Sym}^{k+g}(\underline{L})$ and itself. A homotopy $\hat{y} : ([0,1] \times [0,1], [0,1] \times \{0,1\}) \to (X, \mathrm{Sym}^{k+g}(\underline{L}))$ between a Hamiltonian path

$$y : [0,1] \to X$$

with endpoints on $\mathrm{Sym}^{k+g}(\underline{L})$ and $\mathbf{x}$ is going to be called a **capping**. By [21], Lemma 4.10, the image of the Hurewicz morphism $\pi_2(X, \mathrm{Sym}^{k+g}(\underline{L})) \to H_2(X, \mathrm{Sym}^{k+g}(\underline{L}))$ is freely generated by s homology classes, $u_1, \ldots, u_s$, where s is the number of connected components in $\Sigma_g \backslash \bigcup L_i$. In the case we are interested about there will be $k+1$ connected components and all of them, except for one, will be discs: in this case we have $k+1$ homology classes, $u_1, \ldots, u_{k+1}$, and we get the refined information about their intersections with the diagonal:

$$u_i \cdot \Delta = 2(k_i - 1)$$

where $k_i = 1$ for $i = 1, \ldots, k$ and $k_{k+1} = k$.

Each of these homology classes may now be used to change capping of an orbit using he following method: if u represents any of the u_i above and $\hat{y}$ is a capped Hamiltonian path, we may consider their concatenation $\hat{y}u = \hat{y}\#u$, and the quantity $\omega_X(u) + u \cdot \Delta$ does not depend on the choice of representative u. It makes sense then to define the set of capped orbits modulo equivalence

$$\mathcal{S} = \left\{ \hat{y} | y \in \tilde{\mathcal{S}} \right\} / \sim$$

where $\hat{x} \sim \hat{y} \Leftrightarrow x = y, \omega_X(\hat{x}) + \eta\hat{x} \cdot \Delta = \omega_X(\hat{y}) + \eta\hat{y} \cdot \Delta$. Define now

$$\widetilde{CF}^{\bullet}(\mathrm{Sym}^{k+g}(H), \mathrm{Sym}^{k+g}(\underline{L}); \mathbb{C}) = \bigoplus_{\hat{y} \in \mathcal{S}} \mathbb{C} \cdot [\hat{y}]$$

Recapping gives a $\mathbb{Z}$ action on $\widetilde{CF}^{\bullet}(\mathrm{Sym}^{k+g}(H), \mathrm{Sym}^{k+g}(\underline{L}); \mathbb{C})$, so that we can see it as a $\mathbb{C}[T, T^{-1}]$-module: the Floer complex we are interested in is going to be the tensor product

$$CF^{\bullet}(H, \underline{L}; \mathbb{C}) = \widetilde{CF}^{\bullet}(\mathrm{Sym}^{k+g}(H), \mathrm{Sym}^{k+g}(\underline{L}); \mathbb{C}) \otimes_{\mathbb{C}[T,T^{-1}]} \Lambda$$

with the Novikov field Λ

$$\Lambda = \mathbb{C}[[T]][T^{-1}] = \left\{ \sum_{i=0}^{\infty} a_i T^{b_i} | a_i \in \mathbb{C}, b_i \in \mathbb{Z}, b_i < b_{i+1} \right\}$$

The differential on $CF^{\bullet}(H, \underline{L}; \mathbb{C})$ as customary is defined by a count of holomorphic curves with Lagrangian boundary conditions: fix two Hamiltonian paths $y_i : [0,1] \to X, i = 0, 1$ with boundary conditions on $\mathrm{Sym}^{k+g}\underline{L}$ and a time-dependent almost complex structure on X. We consider strips $u : \mathbb{R} \times [0,1] \to X$ which satisfy the following constraints:

$$\begin{cases} u(s,0),\ u(s,1) \in \mathrm{Sym}^{k+g}(\underline{L}) \\ \lim_{s \to -\infty} u(s,t) = y_0(t),\ \lim_{s \to +\infty} u(s,t) = y_1(t) \\ (\partial_s + J_t \partial_t - X_H) u(s,t) = 0 \end{cases} \tag{1.18}$$

Remark 1.3.29. *Given a holomorphic structure J on Σ_g, one induces naturally a holomorphic structure on X, denoted J_X.*

Solutions to these equations belonging in moduli spaces of virtual dimension 0 (after holomorphic reparametrisation of $\mathbb{R}+i[0,1]$) are going to be regular, so that a differential can be well defined, and it can be proved that $\partial^2 = 0$. Along these curves, the quantity

$$\mathcal{A}_H^\eta(\hat{y}) := \int_0^1 \mathrm{Sym}H_t(y(t))\,dt - \int_{[0,1]\times[0,1]} \hat{y}^*\omega_X - \eta[\hat{y}]\cdot\Delta \tag{1.19}$$

strictly decreases: $\mathcal{A}_H^\eta$ is called the **action**, and $CF^\bullet(H,\underline{L};\mathbb{C})$ is a filtered differential complex. Remark that to make sense of this, we need to observe that $\mathcal{A}_H^\eta$ is constant on the equivalence classes in $\mathcal{S}$, and to make the interaction between the $\mathbb{Z}$ action (or equivalently, the action of the formal variable T) and $\mathcal{A}_H^\eta$ explicit: as it is now classical (see for instance [71]) it corresponds to a translation, and given our monotonicity requirement

$$\mathcal{A}_H^\eta([\hat{y}]T) = \mathcal{A}_H^\eta([\hat{y}]) - \lambda$$

Let $HF_a^\bullet(H,\underline{L};\mathbb{C})$ be the homology of the subcomplex

$$CF_a^\bullet(H,\underline{L};\mathbb{C}) = \bigoplus_{[\hat{y}]\in\mathcal{S},\ \mathcal{A}_H^\eta([\hat{y}])<a} \mathbb{C}\cdot[\hat{y}]$$

In analogy with classical theories, we define the **spectrum** $\mathrm{Spec}(H:\underline{L})$ as the action values of capped intersection points: it is a closed and nowhere dense subset of $\mathbb{R}$ ([56]).

Remark 1.3.30. *If $g=0$ the definition of the action functional in Equation 1.19 is an extension of the usual one: the term counting intersections with the diagonal depends on a procedure of inflation of a Perutz-type symplectic form one does in order to achieve monotonicity of the Lagrangian link in the symmetric product, when the circles satisfy the larger definition of η-monotonicity. For details, we refer to [21], Remark 4.22. For the following it is necessary to notice that one makes no normalisation assumptions on the symplectic form in order to define the action as in 1.19.*

This Lagrangian Floer theory comes with its PSS isomorphisms ([21], [75]), and as Λ-vector spaces

$$HF^\bullet(H,\underline{L};\mathbb{C}) \cong H((\mathbb{S}^1)^k;\Lambda)$$

This implies that $HF^\bullet(H,\underline{L};\mathbb{C})$ is non trivial, and that it moreover has a multiplicative structure ([16]), so let $e\in HF^\bullet(H,\underline{L};\mathbb{C})$ be the unit for its product and define

$$c_{\mathrm{Sym}^{k+g}(\underline{L})}(H) = \inf\{a\in\mathbb{R} | e\in \mathrm{Im}(HF_a^\bullet(H,\underline{L};\mathbb{C}) \to HF^\bullet(H,\underline{L};\mathbb{C}))\}$$

and $c_{\underline{L}} := \frac{1}{k} c_{\mathrm{Sym}^{k+g}(\underline{L})}$. One of the fundamental results in [21] is the following Theorem:

Theorem 1.3.31 ([21], Theorem 1.13). *For any η-monotone Lagrangian link $\underline{L}$ on Σ_g, the link spectral invariant*

$$c_{\underline{L}} : \mathscr{C}^\infty([0,1] \times \Sigma_g) \to \mathbb{R}$$

satisfies the following properties:

- *(Spectrality) For any H, $c_{\underline{L}}(H) \in Spec(H : L)$.*
- *(Hofer Lipschitz) For any H, H' Hamiltonians,*

$$\int_0^1 \min_{x\in\Sigma}(H_t(x) - H'_t(x))\,dt \leq c_{\underline{L}}(H) - c_{\underline{L}}(H') \leq \int_0^1 \max_{x\in\Sigma}(H_t(x) - H'_t(x))\,dt$$

- *(Lagrangian Control) If for each $i \in \{1, \dots, k+g\}$, we have $H_t|_{L_i} = s_i(t)$ for time-dependent constant $s_i : [0,1] \to \mathbb{R}$, then*

$$c_{\underline{L}}(H) = \frac{1}{k+g} \sum_{i=1}^{k+g} \int_0^1 s_i(t)dt$$

  and for a general Hamiltonian

$$\frac{1}{k+g} \sum_{i=1}^{k+g} \int_0^1 \min_{x\in L_i} H_t(x)\,dt \leq c_{\underline{L}}(H) \leq \frac{1}{k+g} \sum_{i=1}^{k+g} \int_0^1 \max_{x\in L_i} H_t(x)\,dt$$

- *(Subadditivity) For any H, H', $c_{\underline{L}}(H\#H') \leq c_{\underline{L}}(H) + c_{\underline{L}}(H')$, if $H\#H'(x,t) = H_t(x) + H'_t((\phi_H^t)^{-1}(x))$*
- *(Homotopy Invariance) If H, H' are two normalised Hamiltonians with same time 1 map and which determine the same element in $\widetilde{Ham}(\Sigma, \omega)$, then $c_{\underline{L}}(H) = c_{\underline{L}}(H')$.*
- *(Shift) If $H = H' + s$, for a function $s \in \mathscr{C}([0,1]; \mathbb{R})$, then*

$$c_{\underline{L}}(H) = c_{\underline{L}}(H') + \int_0^1 s(t)\,dt$$

The proof of this Theorem can be found in Section 6.4 of [21].

In the case $g = 0$, we may homogenise the spectral invariants $c_{\underline{L}}$ to find quasimorphisms on $\mathrm{Ham}(\mathbb{S}^2, \omega)$: if $\tilde{\varphi} \in \widetilde{\mathrm{Ham}}(\mathbb{S}^2, \omega)$ let

$$\mu_{\underline{L}}(\tilde{\varphi}) = \lim_{n\to\infty} \frac{c_{\underline{L}}(\tilde{\varphi}^n)}{n}$$

and since $\pi_1(\mathrm{Ham}(\mathbb{S}^2, \omega))$ is finite, $\mu_{\underline{L}}$ only depends on the time 1 map. We can then formulate the following theorem, proved in Section 7 of [21].

Theorem 1.3.32 ([21], Theorems 7.6, 7.7)**.** *$\mu_{\underline{L}} : Ham(\mathbb{S}^2) \to \mathbb{R}$ satisfies the following properties:*

- *$\mu_{\underline{L}}$ only depends on the monotonicity constant η and on the number of components of the Lagrangian link $\underline{L}$: we will then write $\mu_{k,\eta}$ for the quasimorphism associated to any η-monotone Lagrangian link of k components.*
- *The differences $\mu_{k,\eta} - \mu_{k',\eta'}$ are $\mathscr{C}^0$-continuous.*

Moreover, the properties in Theorem 1.3.31 translate to the following ones for the $\mu_{k,\eta}$:

- *(Hofer Lipschitz) $|\mu_{k,\eta}(\varphi) - \mu_{k,\eta}(\psi)| \leq d_H(\varphi, \psi)$*
- *(Lagrangian Control) If $\varphi = \phi^1_H$ for a mean normalised Hamiltonian H such that $H_t|_{L_i} = s_i(t)$,*

$$\mu_{k,\eta}(\varphi) = \frac{1}{k}\sum_{i=1}^{k}\int_0^1 s_i(t)\,dt$$

  and in general, if H is mean-normalised but not necessarily a time dependent constant on the Lagrangian link:

$$\frac{1}{k}\sum_{i=1}^{k}\int_0^1 \min_{x\in L_i} H_t(x)\,dt \leq \mu_{k,\eta}(\varphi) \leq \frac{1}{k}\sum_{i=1}^{k}\int_0^1 \max_{x\in L_i} H_t(x)\,dt$$

- *(Support Control) If $\varphi = \phi^1_H$ where $Supp(H) \subset \mathbb{S}^2 \setminus \bigcup_i L_i$, then*

$$\mu_{k,\eta}(\varphi) = -Cal(\varphi)$$

As we are going to work with spheres not necessarily of area 1, say a, we shall write $\mu^a_{k,\eta}$ for the quasimorphisms on $\mathrm{Ham}(\mathbb{S}^2(a))$.

Remark 1.3.33. *The properties listed in Theorem 1.3.31 are clearly interrelated, for instance (Shift) is an immediate consequence of (Hofer Lipschitz). The same is true for Theorem 1.3.32, and one remark is needed: for the (Support Control) property we are assuming that the symplectic volume of $\mathbb{S}^2$ to be 1. Indeed, this properties follows from the (Shift) property of $c_{\underline{L}}$, the definition of $\mu_{k,\eta}$ and its (Lagrangian Control) property: to compute $\mu_{k,\eta}(\phi^1_H)$ we need to normalise H, and this operation in general consists in defining H' as follows*

$$H'_t(x) = H_t(x) - \frac{1}{Area(\mathbb{S}^2)}\int_{\mathbb{S}^2} H_t\omega$$

and (Support Control) reads $\mu^{Area(\mathbb{S}^2)}_{k,\eta}(\phi^1_H) = -\frac{1}{Area(\mathbb{S}^2)}Cal(\phi^1_H)$. This fact is not essential but explains the relatively convoluted definition of the quasimorphism appearing in the Appendix B.

Remark 1.3.34. *In the case $g \geq 1$ the spectral invariants do not have the quasimorphism property, but they are shown to be local quasimorphisms in [47].*

1.3.4 Hamiltonian Floer Homology

In this Section we assume that $\pi_2(M) = 0$. Suppose that $\varphi \in \mathrm{Ham}_c(M)$ is non degenerate[3]. Under these hypotheses, one can define the Floer complex associated to φ the following way. Choose an almost complex structure $J \in \mathscr{J}_c(M)$ compatible with the symplectic form (if M is non compact we also have to assume that J behaves well at infinity). The generators of the group are contractible fixed points of φ, and we take their integral linear combinations:

$$CF(H, J; \mathbb{Z}) := \bigoplus_{x \in \mathrm{Fix}(\varphi)} \mathbb{Z} \cdot x$$

This complex is graded by the Conley-Zehnder index. The differential of $CF(H, J; \mathbb{Z})$ counts perturbed J-holomorphic cylinders with asymptotic conditions. Such maps are smooth functions

$$u : \mathbb{R}_s \times \mathbb{S}^1_t \to M$$

solving the Floer equation

$$\partial_s u + J(u)(\partial_t u - X_H(u)) = 0 \tag{1.20}$$

We denote by $\tilde{\mathcal{M}}(x_-, x_+; J)$ the set of all cylinders u satisfying to the previous conditions, and such that are moreover asymptotic to x_- and x_+:

$$x_\pm(t) = \lim_{s \to \pm\infty} t(s, t)$$

where the convergence is uniform in t, and all derivatives in s tend to zero. A map u defined on the cylinder, satisfying Floer equation and asymptotic to Hamiltonian orbits is called a Floer cylinder. Remark that $\tilde{\mathcal{M}}(x_-, x_+; J)$ has an obvious $\mathbb{R}$-action, given by translation along the variable s. Denote by $\mathcal{M}(x_-, x_+; J)$ the quotient of this action. For a generic choice of J, if $CZ(x_-) - CZ(x_+) \leq 2$, the set $\mathcal{M}(x_-, x_+; J)$ is in fact an oriented smooth manifold of dimension $CZ(x_-) - CZ(x_+) - 1$. If $CZ(x_-) - CZ(x_+) = 1$, it is moreover compact, i.e. a finite set of points with signs. We define the differential of $CF(H; \mathbb{Z})$ by

$$\partial x_- := \sum_{CZ(x_+) = CZ(x_-) - 1} \#\mathcal{M}(x_-, x_+; J) x_+$$

The fact that $\partial^2 = 0$ is a deep theorem, and is obtained from the analysis of $\mathcal{M}(x_-, x_+; J)$ when $CZ(x_-) - CZ(x_+) = 2$: in this case it is an 1-dimensional open manifold which may be compactified adding "broken trajectories". This gives a 1-dimensional oriented compact manifold with boundary $\overline{\mathcal{M}}(x_-, x_+; J)$, and the boundary is described by the identity

$$\partial \overline{\mathcal{M}}(x_-, x_+; J) = \coprod_{CZ(x_-) < CZ(y) < CZ(x_+)} \mathcal{M}(x_-, y; J) \times \mathcal{M}(y, x_+; J)$$

[3] Recall that if M is non compact or has non empty boundary we have to perturb φ to achieve non degeneracy.

The complex comes with a filtration: given a loop on the manifold, we consider a capping

$$\hat{x} : \mathbb{D} \to M,\ x(e^{2\pi i t}) = x(t)$$

and define the symplectic action

$$\mathcal{A}_H : \mathscr{C}^\infty(\mathbb{S}^1; M) \to \mathbb{R},\ \mathcal{A}_H(x) := \int_{\mathbb{S}^1} H_t(x(t))\, dt - \int_{\mathbb{D}} \hat{x}^* \omega$$

The Floer complex is morally the Morse complex of the action functional. In particular, the action decreases along the curves we use to define the differential, and extending the definition of $\mathcal{A}_H$ to chains

$$\mathcal{A}_H \left(\sum_i \lambda_i x_i \right) := \max_{i | \lambda_i \neq 0} \mathcal{A}_H(x_i)$$

we obtain the inequality

$$\mathcal{A}_H(\partial x) < \mathcal{A}_H(x)$$

The subgroup

$$CF^\lambda(H, J; \mathbb{Z}) := \bigoplus_{x \in \mathrm{Fix}(\varphi), \mathcal{A}_H(x) < \lambda} \mathbb{Z} \cdot x$$

is therefore a subcomplex. We denote by $HF(H, J; \mathbb{Z})$ the homology of $CF(H, J; \mathbb{Z})$, and by $HF^\lambda(H, J; \mathbb{Z})$ that of the subcomplex filtered by action.

Floer homology is invariant, in the following sense: given two pairs (H, J), (H', J') of regular (i.e. generic) Floer data, there is a homotopy connecting them, and inducing chain morphisms (called "continuation maps")

$$CF(H, J; \mathbb{Z}) \to CF(H', J'; \mathbb{Z})$$

These maps turn out to be quasi-isomorphisms. In particular, the homology $HF(H, J; \mathbb{Z})$ depends neither on the Hamiltonian nor on the almost complex structure we have chosen. Furthermore, if M is compact the Floer homology is isomorphic to the singular homology of the manifold, as computed by Morse theory.

The action filtration also behaves well under continuation maps, in analogy to what happens in the case of generating functions, see (1.16). For two Hamiltonians H and H' denote by

$$\mathcal{E}^+(H' - H) = \int_0^1 \max_{x \in M} (H'_t(x) - H_t(x))\, dt$$

If Φ is a continuation maps from H to H' then for any generator x of $CF(H; \mathbb{Z})$

$$\mathcal{A}_{H'}(\Phi(x)) \leq \mathcal{A}_H(x) + \mathcal{E}^+(H' - H)$$

which means that for all $\lambda \in \mathbb{R}$, Φ restricts to its filtered version

$$\Phi : CF^\lambda(H; \mathbb{Z}) \to CF^{\lambda + \mathcal{E}^+(H' - H)}(H'; \mathbb{Z})$$

The underlying philosophy is that, while the homology of the complex as a whole does not carry information about the specific diffeomorphism at hand, the filtration is a finer invariant. It may be in fact used to perform Hofer measurements: without going into the details, taking the homology of the subcomplexes filtered by maximal action, one finds a persistence module. The space of persistence modules is endowed with a distance, called interleaving distance. The function sending a diffeomorphism to its persistence module turns out to be Lipschitz with respect to the Hofer and interleaving distances; a reference is [59].

Higher homology operations Floer theory comes with higher operations. They are defined counting moduli spaces of curves satisfying a Floer-type equation as well. They were first defined by Schwarz in his PhD thesis [64], and some technical generalisations and improvements are for instance contained in [1], [62] and [25].

Let $\varphi \in \mathrm{Ham}_c(M)$ be non degenerate, let $p \in \mathbb{N}$ and assume that φ^p is also non degenerate. Fix a generating Hamiltonian H for φ, and $H^{\#p}$ its p-th iterate. The p-pair of pants product is a linear map

$$CF(H, J; \mathbb{Z})^{\otimes p} \to CF(H^{\#p}, J; \mathbb{Z})$$

The moduli spaces it counts are defined the following way: let S_p be a model for the pair of pants with p legs, let us assume it is a $p+1$ times punctured 2-dimensional sphere. Among the punctures, exactly one will be called “positive”, the other being “negative” Due to the lack of global coordinates for TS_p one cannot just write Floer equation globally as in the case of the Floer differential. A way to go around this problem is to choose cylindrical ends for S_p: this operation amounts to defining a biholomorphism between a negative half-cylinder

$$(-\infty, -R] \times \mathbb{S}^1 \ \ (R > 0)$$

and a neighbourhood of a puncture, for the p negative punctures, and a biholomorphism between a positive half cylinder

$$[R, +\infty) \times \mathbb{S}^1 \ \ (R > 0)$$

and a neighbourhood of the only positive puncture. We assume cylindrical ends of different punctures to be disjoint. On each cylindrical end we can now write Floer equation, so

$$\mathcal{M}(x_1, \ldots, x_p; y; J)$$

is defined to be the set of smooth maps

$$u : S_p \to M$$

which at the i-th negative puncture tend to x_i (1-periodic orbit of φ), at the positive puncture are asymptotic to y (closed orbit of φ, of period dividing p), satisfy Floer equation on the cylindrical ends and are J-holomorphic elsewhere. The last two conditions are made to be compatible using a partition of the identity on S_p. For the good definition of the operation, i.e. for the moduli spaces to be smooth manifolds of the right dimension, we still need transversality. We mention that it can be achieved (see the references given above) via what is called a domain-dependent perturbation: whenever needed, we require u as above to satisfy a so perturbed Floer equation on the cylindrical ends.

Counting elements in $\mathcal{M}(x_1, \ldots, x_p; y; J)$ defines an operation between chain complexes

$$CF_{k_1}(H, J; \mathbb{Z}) \otimes \cdots \otimes CF_{k_p}(H, J; \mathbb{Z}) \to CF_{2(1-p)+\sum_{i=1}^{p} k_i}(H^{\#p}, J; \mathbb{Z})$$
$$x_1 \otimes \cdots \otimes x_p \mapsto \sum_{y} \#\mathcal{M}(x_1, \ldots, x_p; y; J) \cdot y$$

The sum runs on the fixed points y of φ^p of correct Conley-Zehnder index.

What we shall need next is a holomorphicity Lemma proved by Fabert. Fix a Hamiltonian H and an almost complex structure J on M. We paraphrase the Lemma as follows.

Lemma 1.3.35 (Fabert, [25]). *Let $\pi : S_p \to \mathbb{R} \times \mathbb{S}^1$ be a holomorphic cover of the cylinder. Let $u : S_p \to M$ be smooth. Then there exists an almost complex structure $\tilde{J}$ on $\mathbb{R} \times \mathbb{S}^1 \times M$ such that, given a pair of pants u which is negatively asymptotic to $x_1, \ldots, x_p$ and positively asymptotic to y, $u \in \mathcal{M}(x_1, \ldots, x_p; y, J)$ if and only if the map*

$$(\pi, u) : S_p \to \mathbb{R} \times \mathbb{S}^1 \times M$$

is $\tilde{J}$ holomorphic.

1.3.5 Siefring product of Punctured Holomorphic Curves

We now recall the basic properties of the intersection product of punctured holomorphic curves in 4-manifolds as defined by Siefring [66] (but a good source is also [74], and we are going to heavily draw from there).

Remark 1.3.36. *Since our definition of linking number differs from that of [67] and [74], we shall have to insert normalising factors of $\frac{1}{2}$ in some of the definitions below, to maintain consistency.*

The Siefring product $*$ of two asymptotically cylindrical maps (maps defined on a punctured Riemann surface with values in a symplectisation and asymptotic conditions near the punctures) is an integer which generalises the intersection product for pseudo-holomorphic maps in 4-manifolds with compact domain. In particular, the Siefring product is a homotopy invariant in a class of asymptotically cylindrical maps with fixed asymptotics. We refer to [66] and [74] for details of the definitions. Since we are going to apply this to curves appearing in the definition of Hamiltonian Floer homology, we cast everything already in this language: we shall focus on curves of genus 0, with p negative punctures and 1 positive one, embedded in a symplectisation and asymptotic to simply covered Hamiltonian orbits. This way, the discussion greatly simplifies and we do not need to define asymptotically cylindrical maps with greater generality.

Fix $\varphi \in \mathrm{Ham}(\Sigma)$, where Σ is either a closed symplectic surface or $\mathbb{R}^2$ generated by a Hamiltonian H (if $\Sigma = \mathbb{R}^2$, we assume H to be compactly-supported). Fix an ω-compatible almost complex structure J on Σ. The 2-form, not symplectic in general, we consider on $\mathbb{R}_s \times \mathbb{S}^1_t \times \Sigma$ is twisted by the Hamiltonian: if ω is the form we consider on Σ, the one we obtain on the 4-manifold is

$$\Omega := \omega + dt \wedge dH$$

Now fix two cylinders

$$u, v : \mathbb{R}_s \times \mathbb{S}^1_t \to \Sigma$$

satisfying Floer equation (1.20). The graphs of u and v, from now on denoted $\overline{u}$ and $\overline{v}$, are holomorphic for a choice of compatible almost complex structure:

$$\tilde{J}|_{T\Sigma} = J, \;\; \tilde{J}\partial_s = \partial_t + X_H$$

The almost complex structure $\tilde{J}$ is also invariant under s-translation. Such an almost complex structure is said to be "cylindrical". One can check that u and v satisfy the Floer equation if and only if $\tilde{u}$ and $\tilde{v}$ are $\tilde{J}$-holomorphic.

For now we assume $\overline{u}$ and $\overline{v}$ have only finitely many intersections. The Siefring product $\overline{u} * \overline{v}$ will be the sum of two terms: $\overline{u} \cdot \overline{v}$, which counts intersections between the graphs contained in a compact set, and $\iota_\infty(u, v)$, which may be interpreted as a count of "intersections hidden at infinity". What follows is needed to detail the definition of the latter contributions.

We start by fixing a symplectic trivialisation of $T\Sigma$ at the asymptotic orbits of u and v, let τ be this trivialisation. We can now look at the asymptotic Floer operator: in such a trivialisation it takes the form

$$\mathbf{A} = -J_0 \frac{d}{dt} - S$$

where J_0 is the standard complex structure on $\mathbb{R}^2$ and S is a closed path of symmetric 2×2 matrices. It is a Theorem ([31]) that the eigenfunctions of this kind of operators have a well defined linking number (they solve a non-autonomous linear ODE), and moreover eigenfunctions sharing the same eigenvalue have the same linking number. For an eigenvalue $\lambda \in \sigma(\mathbf{A})$ of $\mathbf{A}$ we may then write

$$\mathfrak{lk}^\tau(\lambda) = \frac{1}{2}\mathrm{lk}(f_\lambda, 0), \text{ where } \mathbf{A} f_\lambda = \lambda f_\lambda \text{ and } 0 \text{ is the constant path at } 0$$

This function is increasing, surjective in $\mathbb{Z}$ and every integer has precisely two preimages.

We define then what is called the extremal winding at an orbit γ

$$\alpha^\tau_+(\gamma) = \min\{\, \mathfrak{lk}^\tau(\lambda) | \lambda \in \sigma(\mathbf{A}), \lambda > 0 \,\}$$
$$\alpha^\tau_-(\gamma) = \max\{\, \mathfrak{lk}^\tau(\lambda) | \lambda \in \sigma(\mathbf{A}), \lambda < 0 \,\}$$

and the parity

$$p(\gamma) = \alpha_+^\tau(\gamma) - \alpha_-^\tau(\gamma)$$

which takes values in $\{0, 1\}$. These three quantities are related to the Conley-Zehnder index[4] of γ by the relations

$$-CZ^\tau(\gamma) = 2\alpha_-^\tau(\gamma) + p(\gamma) = 2\alpha_+^\tau(\gamma) - p(\gamma) \tag{1.21}$$

Remark that the Conley-Zehnder index in principle does depend on the trivialisation one chooses, but this was not important in our discussion above since we restricted to the symplectically aspherical case.

Denote by u_+, v_+ (resp. by u_-, v_-) the orbits which u and v are positively (resp. negatively) asymptotic to. We set now

$$\iota^\tau(u, \pm\infty; v \pm \infty) = 0$$

if $u_\pm \neq v_\pm$. Otherwise we do the following: assume $u(+\infty, \cdot) = v(+\infty, \cdot) = \gamma$, parameterise $u(s, \cdot) = v(s, \cdot)$ for $s >> 0$ using an exponential map on $\gamma^* T\Sigma$. Call

$$h_v, h_u : [s_0, +\infty) \times \mathbb{S}^1 \to \gamma^* T\Sigma$$

such expressions for u and v respectively. Remember that the τ we defined above restricts to a trivialisation for $\gamma^* T\Sigma$ (such τ is in fact a choice of trivialisation at every asymptotic orbit of our pairs of pants). The number $\iota_\infty^\tau(u, \pm\infty; v, \pm\infty)$ is defined as

$$\iota_\infty^\tau(u, \pm\infty; v, \pm\infty) = \mp\frac{1}{2}\mathrm{lk}^\tau(h_u(s, \cdot) - h_v(s, \cdot)) \tag{1.22}$$

The term on the right is the linking number with 0 of a $\mathbb{S}^1$-family of nonzero complex numbers: we have a braid with two strands in $\mathbb{C}$, and we compute the linking number as defined at the beginning of this Chapter. The expression in the right hand side of (1.22) is well-defined because of our hypothesis: u and v have only finitely many intersections, so in particular

$$h_u(s, \cdot) - h_v(s, \cdot) \neq 0$$

for $|s| >> 0$. We highlight the fact that ι_∞^τ coincides with the asymptotic linking number only at negative punctures, it is its opposite at the positive ones. We define $\iota_\infty^\tau(u, v)$ to be the sum of the contributions of linking numbers at positive and negative infinity:

$$\iota_\infty^\tau(u, v) := \iota_\infty^\tau(u, -\infty; v, -\infty) + \iota_\infty^\tau(u, +\infty; v, +\infty)$$

Now, by definition of $\alpha_\pm^\tau(\gamma)$, $\iota_\infty^\tau(u, \pm\infty; v, \pm\infty)$ may be a priori bounded: since u and v are pseudo-holomorphic, up to an infinitesimal error the maps h_u and h_v coincide with eigenfunctions of the asymptotic Floer operator, multiplied

[4]Our normalisation is the opposite of that in [74] and [66]

by an exponential depending the s-coordinate and asymptotic to 0. Defining the integer

$$\Omega^\tau_\pm(\gamma_1,\gamma_2) := \begin{cases} \mp\alpha^\tau_\mp(\gamma) & \gamma_1=\gamma_2=\gamma \\ 0 & \gamma_1\neq\gamma_2 \end{cases} \tag{1.23}$$

we have the inequality

$$\iota^\tau_\infty(u,\pm\infty;v,\pm\infty) \geq \Omega^\tau_\pm(u(\pm\infty,\cdot),v(\pm\infty,\cdot)) \tag{1.24}$$

So for u,v Floer cylinders with finitely many intersections we set

$$\iota_\infty(u,v) := \iota^\tau_\infty(u,v) - (\Omega^\tau_-(u(-\infty,\cdot),v(-\infty,\cdot)) + \Omega^\tau_+(u(+\infty,\cdot),v(+\infty,\cdot)))$$

and one may check that $\iota_\infty(u,v)$ does indeed not depend on the trivialisation τ. We define the Siefring product of u and v to be

$$\overline{u} * \overline{v} := \overline{u}\cdot\overline{v} + \iota_\infty(u,v) \tag{1.25}$$

If $\overline{u}$ and $\overline{v}$ do not have finitely many intersections, we perturb either cylinder, say $\overline{v}$ by a section of $\overline{v}^*T\Sigma$ (the tangent space of Σ is naturally a subspace of the tangent space of the symplectisation) whose linking number at $\pm\infty$ at both punctures is 0. Let $\overline{v}_\tau$ be the perturbed copy of $\overline{v}$: it has now finitely many intersections with u and we define

$$\overline{u} \bullet_\tau \overline{v} := \overline{u}\cdot\overline{v}_\tau$$

In this case, the Siefring product is defined to be

$$\overline{u} * \overline{v} := \overline{u}\bullet_\tau\overline{v} - (\Omega^\tau_-(u(-\infty,\cdot),v(-\infty,\cdot)) + \Omega^\tau_+(u(+\infty,\cdot),v(+\infty,\cdot))) \tag{1.26}$$

We extend these definitions to the case of curves with several positive and negative punctures, asymptotic to Hamiltonian orbits at each puncture, such as those we use in the definition of Floer products. The context we are going to apply Siefring's theory here will be a bit different from the above, since we are not going to consider curves in $\mathbb{R}_s\times\mathbb{S}^1_t\times\Sigma$ anymore, rather into a branched cover of it. Since the curves will still live in an almost complex cobordism in the sense of [66, Section 2.2] and this is the setting in which Siefring's theory was originally cast, what we stated above can indeed be extended to the case of curves with more than two punctures. In principle we could keep working with pairs-of-pants in the symplectisations, but we would run into two problems. The first is that at punctures curves in $\mathbb{R}\times\mathbb{S}^1\times\Sigma$ the asymptotic orbits might turn out to be multiply covered, and the definitions we shall presently give here should be changed to take this into account. The second is that we would not be able to interpret intersections counting towards $\overline{u}*\overline{v}$ as changes in linking numbers of braids.

If u and v are two pairs-of-pants with p negative punctures (inputs) and 1 positive puncture (output), we see them as maps

$$u,v : \mathbb{S}^2\setminus\{z^1_-,\dots,z^p_-,z_+\}\to\Sigma$$

As before we consider their graphs (embbeded in a branched covering of the symplectisation $\mathbb{R}_s \times \mathbb{S}^1_t \times \Sigma$), and we want to compute their intersections. The asymptotic quantities ι^τ_∞ and $\Omega^\tau_\pm$ still make sense in the case of pairs-of-pants since they just need cylindrical coordinates near the ends for their definition. We define

$$\iota^\tau_\infty(u, z^i_-; v, z^j_-),\ \iota^\tau_\infty(u, z_+; v, z_+)$$

as above, and we have

$$\iota^\tau_\infty(u, v) := \iota^\tau_\infty(u, z_+; v, z_+) + \sum_{i,j=1}^{p} \iota^\tau_\infty(u, z^i_-; v, z^j_-) \tag{1.27}$$

Likewise, we set

$$\iota_\infty(u, v) := \iota^\tau_\infty(u, v) - \left(\Omega^\tau_+(u(z_+, \cdot), v(z_+, \cdot)) + \sum_{i,j=1}^{p} \Omega^\tau_-(u(z^i_-, \cdot), v(z^j_-, \cdot))\right) \tag{1.28}$$

to define $u * v$ formally as above if u and v have finitely many intersections, otherwise we set

$$\overline{u} * \overline{v} := \overline{u} \bullet_\tau \overline{v} - \left(\Omega^\tau_+(u(z_+, \cdot), v(z_+, \cdot)) + \sum_{i,j=1}^{p} \Omega^\tau_-(u(z^i_-, \cdot), v(z^j_-, \cdot))\right) \tag{1.29}$$

We highlight the fundamental property that the Siefring product is a homotopy invariant in each class of asymptotically cylindrical curves; in particular if $\overline{u}$ and $\overline{u'}$ are asymptotically cylindrical maps which are homotopic via a compactly supported homotopy, for any other asymptotically cylindrical map $\overline{v}$ we have the equality

$$\overline{u'} * \overline{v} = \overline{u} * \overline{v}$$

This simple fact is one of the main motivations for the definition of the Siefring product, and a heuristic explanation may be found above, in the discussion after (1.25).

Chapter 2

Hofer Norms on Braid Groups

2.1 Introduction

In this chapter we report the results contained in [51] and in a joint work with Ibrahim Trifa [52]. The results of the former are about the Hofer geometry of $\mathrm{Ham}_c(\mathbb{D})$, while those of the latter will give information about the Hofer geometry of $\mathrm{Ham}_c(\Sigma_{g,p})$ for any $g \geq 0$ and $p \geq 1$. Since the work with Trifa is more general and there is a significant overlap of the techniques we adopt, we report the results and methods directly from [52] with the addition of some material from [51] which is used there. The hope is to avoid redundancies which might make the exposition heavier.

Given a pre-monotone configuration of circles (Definition 2.2.1), we consider the subgroup of Ham fixing it. To every diffeomorphism φ of this kind, we may associate a braid type $b(\varphi)$, which is an element in $\mathcal{B}_{n,g,p}$. We give an estimate from below on the Hofer norm of φ which only depends on the pre-monotone configuration of circles and $b(\varphi)$. We then pushforward the Hofer norm on Ham to obtain a pseudonorm on $\mathcal{B}_k$ (in the case of the disc) or on a proper subgroup of $\mathcal{B}_{n,g,p}$ (for higher genus surfaces). Our results on the geometry of Ham will be pushed forward as well to lower bounds on the induced metrics on braid (sub)groups.

Our main result might be stated as follows:

Theorem 2.1.1. *Let $\underline{L}$ be a pre-monotone configuration of circles on $\Sigma_{g,p}$, and let $\mathcal{B}_{\underline{L}} \subseteq \mathcal{B}_{k,g,p}$ be the image of the braid type function b. There exists a function $\mathfrak{f} : \mathcal{B}_{\underline{L}} \to \mathbb{Z}$, non trivial on the generators of $\mathcal{B}_{\underline{L}}$ and unbounded, such that if φ preserves the link $\underline{L}$ then*

$$\|\varphi\| \geq \frac{1}{2}|\mathfrak{f}(b(\varphi))|$$

For the precise statement, we refer to Theorem 2.2.3.

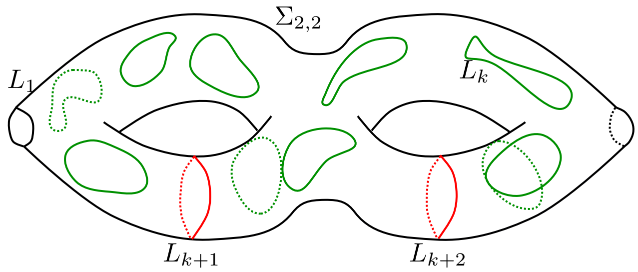


Figure 2.1: This illustration shows what a pre-monotone configuration looks like, without claiming to accurately depict the area conditions. In red we have the non contractible components of the configuration, in green the contractible ones.

2.2 Setup and Main Results

Let $\Sigma_{g,p}$ be a compact oriented surface of genus g with p boundary components, with area form ω normalised so that

$$\int_{\Sigma_{g,p}} \omega = 1$$

Let $\underline{L} = L_1, \dots, L_{k+g}$ be a set of $k+g$ disjoint, embedded circles in $\Sigma_{g,p}$: we call $\underline{L}$ a Lagrangian link.

Definition 2.2.1. *We say that $\underline{L}$ is premonotone if the following conditions are satisfied:*

i) exactly g of the circles in $\underline{L}$ are non contractible;

ii) $\Sigma_{g,p} \setminus \bigcup_{i=1}^{k+g} L_i$ is a disjoint union of k discs $(B_j)_{j=1,\dots,k}$ and a pair of pants B_{k+1} with $p+k+2g-1$ legs;

iii) There exist $\lambda \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$, such that $\lambda = \int_{B_j} \omega$ for $j = 1, \dots, k$.

See Figure 2.1 for a depiction of a configuration satisfying conditions i) and ii).

Given a premonotone Lagrangian link, we denote by $\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$ the subgroup of $\mathrm{Ham}_c(\Sigma_{g,p})$ stabilising the Lagrangian link as a set:

$$\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p}) := \left\{ \varphi \in \mathrm{Ham}_c(\Sigma_{g,p}) | \exists \sigma \in \mathfrak{S}_{k+g}, \varphi(L_i) = L_{\sigma(i)} \right\}$$

To a diffeomorphism in $\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$ we may associate an element of the braid group $\mathcal{B}_{k,g,p}$. This function is defined the following way: choose one base point per contractible circle in $\underline{L}$, denote it by $p_i \in L_i$ for $i = 1, \dots, k$. Any Hamiltonian isotopy $(\varphi_t)_{t\in[0,1]}$ between the identity and $\varphi \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$ provides then a collection of k curves $t \mapsto \varphi_t(p_i)$. Remark that there exists a permutation σ in $\mathfrak{S}_{k+g}$ such that for all i, $\varphi(L_i) = L_{\sigma(i)}$, and contractible circles are mapped

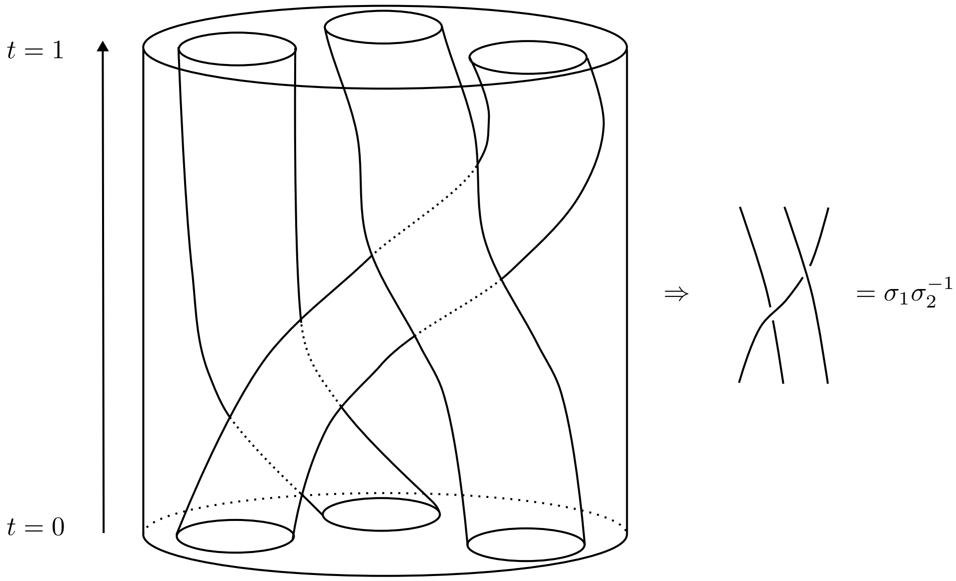


Figure 2.2: The definition of the braid-type function b in the case $\Sigma_{g,p} = \mathbb{D}$.

to contractible circles. For each contractible L_i, choose then a path in $L_{\sigma(i)}$ connecting $\varphi(p_i)$ to $p_{\sigma(i)}$. The concatenation of the curves $t \mapsto \varphi_t(p_i)$ with the respective connecting paths yields an element in $\mathcal{B}_{k,g,p}$. The braid obtained this way does not depend on the choice of the base points, of connecting paths or of the Hamiltonian isotopy (a proof of this fact is included later). We thus have defined a groups homomorphism

$$b : \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p}) \to \mathcal{B}_{k,g,p}$$

Denote by $\mathcal{B}_{\underline{L}}$ the image of b (it will be described in detail in Section 2.3.3).

Remark 2.2.2. *We choose to require in this definition that the circles $L_1, \ldots, L_k$ bound discs of same area. Allowing for discs of different areas might result in a slight generalisation, but since we are considering Hamiltonian maps (which in particular preserve areas), if the configuration is not pre-monotone we are not going to be able to produce any braid in $\mathcal{B}_{n,g,p}$ as braid type of a Hamiltonian diffeomorphism. If however the configuration is pre-monotone and $\Sigma_{g,p} = \mathbb{D}$, the morphism $Ham_{\underline{L}}(\mathbb{D}, \omega) \to \mathcal{B}_k$ is easily seen to be surjective: if $k = 2$, up to symplectomorphism of $\mathbb{D}$ the two discs are conjugated by a half-turn, which corresponds to a generator of $\mathcal{B}_2$. If $k \geq 3$, to represent any generator of $\mathcal{B}_k$ it suffices to consider a disc containing the two circles corresponding to the two strands in play, and then remark that we can apply what said in the case $k = 2$. If $g \geq 1$, the braid type function b cannot be surjective. Indeed, any non contractible circle in $\underline{L}$ is Hamiltonianly non displaceable, and therefore has to be mapped to itself by any element in $Ham_{\underline{L}}(\Sigma_{g,p})$. See Section 2.3.3 for details.*

Our main result is the following:

Theorem 2.2.3. *There exists a family* $(\mathfrak{f}_{(v_1,v_2)})_{v_i \in V}$ *of group homomorphisms* $\mathcal{B}_{\underline{L}} \to \mathbb{R}$ *indexed on the square of a p-dimensional simplex* V *(defined in Section 2.4.2) such that for every* $v_i \in V$ *and* $\varphi, \psi \in Ham_{\underline{L}}(\Sigma_{g,p})$,

$$d_H(\varphi,\psi) \geq \frac{1}{2}|\mathfrak{f}_{(v_1,v_2)}(b(\varphi\psi^{-1}))| \tag{2.1}$$

In particular, we have the following estimate for the Hofer norm:

$$\|\varphi\| \geq \frac{1}{2} \sup_{(v_1,v_2)\in V^2} |\mathfrak{f}_{(v_1,v_2)}(b(\varphi))| \tag{2.2}$$

If we define the following function on $\mathcal{B}_{\underline{L}}$:

$$\mathfrak{f}(g) = \max_{(v_1,v_2)\in V^2} |\mathfrak{f}_{(v_1,v_2)}(g)| \tag{2.3}$$

In the case of the generators σ_j, a_i, $b_i^{-1}a_ib_i$ *we then obtain the explicit values:*

$$\mathfrak{f}(\sigma_j) = \frac{1}{2}\mathfrak{f}(a_i) = \frac{1}{2}\mathfrak{f}(b_i^{-1}a_ib_i) = \frac{1}{2(k+g)}\frac{(k+1)\lambda - 1}{k+2g-1}$$

Remark 2.2.4. *We compute in (2.17) explicit values for the maximum in 2.3*

Remark 2.2.5. *This result seems counter-intuitive when compared to a result of Khanevsky's contained in [39]. In that paper, Khanevsky defines the notion of homological trajectory of a Hamiltonian diffeomorphism defined on a surface and fixing a given disc. He then proceeds to prove that, whenever the genus of the surface is positive, there exists a constant only depending on the disc such that one can realise any trajectory with a Hamiltonian diffeomorphism of Hofer energy lower than this threshold. In fact, it is easily seen that such diffeomorphisms cannot stabilise any non contractible component of the Lagrangian link* $\underline{L}$*, therefore they do not appear in our treatment. If anything, this shows that to find a result like ours, the constraint of fixing a number of non contractible circles equal to the genus is in fact minimal (otherwise it is possible to construct those diffeomorphisms, realising arbitrarily complex braids with bounded Hofer energy).*

In the case of the disc, we in fact find a quasimorphism that is sensitive to the linking number of braids (this is contained in [51]).

Theorem 2.2.6. *Let* $\underline{L}$ *be a pre-monotone Lagrangian link in* $\mathbb{D}$ *with* k *components bounding discs of area* $\lambda \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$*. There exists a Hofer 2-Lipschitz homogeneous quasimorphism* $Q_k : Ham_c(\mathbb{D},\omega) \to \mathbb{R}$ *such that if* $\varphi \in Ham_{\underline{L}}(\mathbb{D},\omega)$ *has associated braid type* $b(\varphi)$*, then*

$$Q_k(\varphi) = \frac{1}{2k}\frac{(k+1)\lambda - 1}{2(k-1)}\mathrm{lk}(b(\varphi))$$

Remark 2.2.7. *There exists in fact a linear independent family of quasimorphisms with the feature that restricted to $Ham_{\underline{L}}(\mathbb{D},\omega)$ they are proportional to the linking of the braid associated to the diffeomorphism; the quasimorphism in Theorem 2.2.6 happens to be the one with the largest proportionality constant.*

We prove Theorems 2.2.3 and 2.2.6 in Section 2.4.2.

We now define a norm on $\mathcal{B}_{\underline{L}}$. For any $a \in \mathcal{B}_{\underline{L}}$ we set

$$\|a\|_{\underline{L}} := \inf_{\varphi \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p}), b(\varphi)=a} \|\varphi\| \tag{2.4}$$

It is easy to check that $\|\cdot\|$ is a pseudonorm on $\mathcal{B}_{\underline{L}}$.

A corollary of Theorem 2.2.3 is an estimate from below of $\|\cdot\|_{\underline{L}}$:

Corollary 2.2.8. *Let $a \in \mathcal{B}_{\underline{L}}$. Then, if $\mathfrak{f}$ is the function from Theorem 2.2.3,*

$$\|a\|_{\underline{L}} \geq \frac{1}{2} \cdot |\mathfrak{f}(a)| \tag{2.5}$$

As in [51], this is not enough to conclude non degeneracy as soon as $k+g \geq 2$ (as soon as the genus is positive, there are no interesting cases we can consider for $k+g \leq 2$). We may however adapt a proof by Chen contained in [15] to show that

Theorem 2.2.9. *For a pre-monotone $\underline{L}$, the pseudonorm $\|\cdot\|_{\underline{L}}$ is non degenerate.*

Chen had in particular proved this fact for braids on the disc.

Recall the existence of the real-valued Calabi morphism on $\mathrm{Ham}_c(\Sigma_{g,p})$ (for a definition, see for instance [49, Section 10.3]). An immediate corollary of this result is the existence of Hamiltonian diffeomorphisms in the kernel of Calabi with nonzero asymptotic Hofer norm.

For any $\varphi \in \mathrm{Ham}_c(\Sigma_{g,p})$, we define its asymptotic Hofer pseudonorm by

$$\|\varphi\|_\infty := \lim_n \frac{\|\varphi^n\|}{n}$$

We may now state the following elementary consequence of Theorem 2.2.3:

Corollary 2.2.10. *Let $\varphi \in Ham_{\underline{L}}(\Sigma_{g,p})$. Then*

$$\|\varphi\|_\infty \geq \frac{1}{2} \cdot |\mathfrak{f}(b(\varphi))|$$

Moreover, any braid type in the image of b may be realised by a diffeomorphism in the kernel of Calabi.

Proof. Since $\mathfrak{f}_{(v_1,v_2)}$ and b are both morphisms of groups for all $v_i \in V$, the only statement which is not entirely trivial is the one about the kernel of Calabi. This is easily seen, since Calabi is a morphism of groups, and we may compose any element in $\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$ by a diffeomorphism realising the trivial braid and arbitrary Calabi. □

2.3 Technical Preliminaries

2.3.1 A proof of monotonicity

In this section we are going to prove that, given a pre-monotone (Definition 2.2.1) Lagrangian link $\underline{L}$ on a closed surface Σ_g, and a Hamiltonian H on Σ_g, the Quantitative Heegaard-Floer Homology of the pair $(\underline{L}, H)$ is well defined. We achieve it showing Proposition 2.3.1 and applying the arguments contained in [21]. Our Proposition is needed since in [21] they need more non contractible link components to prove a monotonicity result for the Lagrangian (they use links such that the closure of each connected component of the complement is planar). The homology also turns out to be nonzero: this follows from an application of the Künneth formula (together with the independence of the Homology on the Floer data) developed by Trifa and Mak in [47].

We assume that $\int_{\Sigma_g} \omega = A$, A not necessarily being 1. In the following, we split the circles in $\underline{L}$ in two classes: the contractible and the non contractible ones. We assume that $L_1, ..., L_k$ $(k \geq 2)$ bound disjoint discs $B_1, ..., B_k$ of area $\lambda \in [\frac{A}{k+1}, \frac{A}{k})$, and that $\alpha_1, ..., \alpha_g$ $(\alpha_i := L_{k+i})$ be meridians for each handle of Σ_g. This way, we have $\underline{L} = L_1 \cup ... \cup L_k \cup \alpha_1 \cup ... \cup \alpha_g$ and $X := \mathrm{Sym}^{k+g}(\Sigma_g)$. Let B_{k+1} be the only connected component of $\Sigma_g \setminus \underline{L}$ that is not a disc, and let A_{k+1} be its area. Let η be the real number satisfying

$$A_{k+1} + 2\eta(k + 2g - 1) = \lambda$$

Then η can be recovered by the formula

$$\eta = \frac{\lambda - A_{k+1}}{2(k + 2g - 1)} = \frac{(k + 1)\lambda - A}{2(k + 2g - 1)}$$

and in particular is non negative.

Proposition 2.3.1. *For all $[u]$ in the image of $\pi_2(X, Sym^{k+g}(\underline{L})) \to H_2(X, Sym^{k+g}(\underline{L}))$ one has*

$$\omega_X([u]) + \eta\Delta \cdot [u] = \frac{\lambda}{2}\mu([u])$$

Proof. For $1 \leq i \leq k + 1$, let $\overline{B}_i$ be the closure of B_i in Σ_g. Let k_i be the number of boundary components of $\overline{B}_i$. For each i, fix a point a_i in B_i, and let X_{a_i} be the projection of $\Sigma_g^{k+g-1} \times \{a_i\}$ in $\mathrm{Sym}^{k+g}(\Sigma_g)$. In [21, Section 4.5], it is explained how, when $\overline{B}_i$ is planar, one can construct a disc class $[u_i]$ in $H_2(X, \mathrm{Sym}\underline{L})$, which satisfies $[u_i] \cdot X_{a_j} = \delta_{i,j}$, $[u_i] \cdot \Delta = k_i - 1$, and $\mu([u_i]) = 2$. Here, the $\overline{B}_i$'s for $1 \leq i \leq k$ are discs, therefore we can apply this construction to get k classes satisfying $\omega_X([u_i]) + \eta\Delta \cdot [u_i] = \lambda + 0 = \frac{\lambda}{2}\mu([u_i])$.

However, $\overline{B}_{k+1}$ is not a planar domain (it is equal to the whole surface minus k disjoint discs). But we can still apply a similar construction:

Let $\widehat{D} := \overline{B}_{k+1} \sqcup \coprod_{1 \leq j \leq g} D_j$, where the D_j's are copies of the closed unit disc D in $\mathbb{C}$. Let $\pi_{\widehat{D}} : \widehat{D} \to D$ be a $(k + g)$-fold simple branched covering, such that

$\pi_{\widehat{D}}|_{D_j}$ is a biholomorphism for all j, and $\pi_{\widehat{D}_i}|_{\overline{B}_{k+1}}$ is a topological k-fold simple branched cover, such that a_{k+1} is not a branched point. Since $k \geq 2$, such a branched cover always exists; one can construct it using techniques presented in John Etnyre's lecture notes [24]. Let $v_{k+1} : \widehat{D} \to \Sigma_g$ be a map whose restriction to $\overline{B}_{k+1}$ is the identity, and which sends D_j to a point in α_j. Then, by tautological correspondence, we get a map $u_{k+1} : (D, \partial D) \to (X, \mathrm{Sym}\underline{L})$, defined by $u_{k+1}(z) = v_{k+1}(\pi_{\widehat{D}}^{-1}(z))$. The class $[u_{k+1}]$ also satisfies $[u_{k+1}] \cdot X_{a_i} = v_{k+1} \cdot a_i = \delta_{i,k+1}$ for $1 \leq i \leq k+1$. Therefore, the proof of [21, Lemma 4.10] goes through, and we get that the image of $\pi_2(X, \mathrm{Sym}\underline{L}) \to H_2(X, \mathrm{Sym}\underline{L})$ is freely generated by $\{[u_i]\}_{i=1}^{k+1}$, and that the image of $\pi_2(X) \to H_2(X, \mathrm{Sym}\underline{L})$ is freely generated by $\sum_{i=1}^{k+1} [u_i]$.

Since the $[u_i]$ satisfy the equation $\omega_X([u_i]) + \eta\Delta \cdot [u_i] = \frac{\lambda}{2}\mu([u_i])$ for $1 \leq i \leq k$, it only remains to show it for $[u_{k+1}]$.

We start by computing $[u_{k+1}] \cdot \Delta$ in a similar fashion as in [21]. Let $\pi : \Sigma_g \to S^2$ be a topological $(k+g)$-fold simple branched covering of the sphere. Let $u : S^2 \to X$ be the map tautologically corresponding to the pair (π, id_{Σ_g}). Then, since $[u]$ is in the image of $\pi_2(X) \to H_2(X, \mathrm{Sym}\underline{L})$, there exists an integer c such that $[u] = c\sum_{i=1}^{k+1}[u_i]$. Since $[u] \cdot X_{a_i} = 1$ for all i, we get that $c = 1$. Therefore, $[u] \cdot \Delta = (\sum_{i=1}^{k+1}[u_i]) \cdot \Delta = [u_{k+1}] \cdot \Delta$. Since id_{Σ_g} is injective, there is a one-to-one correspondence between branched point of π and points of S^2 whose image by u lies in Δ. Moreover, since the branched points are simple, we have that $[u] \cdot \Delta$ is actually equal to the number of branched points of π. Therefore, the Riemann-Hurwitz formula implies that $2 - 2g = 2(k+g) - [u] \cdot \Delta$, and we finally get $[u_{k+1}] \cdot \Delta = 2(k + 2g - 1)$. It only remains to compute the Maslov index of $[u_{k+1}]$:

$$\mu([u_{k+1}]) = \mu\left([u] - \sum_{i=1}^{k}[u_i]\right) = 2\langle c_1(TX), [u]\rangle - 2k$$

Let v be the Abel-Jacobi map, from $\mathrm{Sym}^d(\Sigma_g)$ to its Jacobian variety J, which is isomorphic to the $2g$-dimensional torus T^{2g}. According to [5, Chapter VII, Section 5], there is a class θ in $H^2(J)$ and a point q in Σ_g such that:

$$c_1(T\mathrm{Sym}^d\Sigma_g) = (d - g + 1)PD(X_q) - v^*\theta$$

Now, in our case, $d = k + g$, and as above $[u]$ is a generator of the image of $\pi_2(X) \to H_2(X, \mathrm{Sym}\underline{L})$. Since $J = T^{2g}$ is aspherical, $v_*[u]$ vanishes, and

$$\mu([u_{k+1}]) = 2\langle c_1(TX), [u]\rangle - 2k = 2(k+1)[u] \cdot X_q - 2k = 2$$

where the last equality comes from the fact that $[u] \cdot X_q = id_{\Sigma_g} \cdot q = 1$ when q is not a branched point of π, which we can assume by perturbing π if necessary. As claimed, we get

$$\omega_X([u_{k+1}]) + \eta\Delta \cdot [u_{k+1}] = A_{k+1} + 2\eta(k + 2g - 1) = \lambda = \frac{\lambda}{2}\mu([u_i])$$

□

Remark 2.3.2. *One can use the results of this section to show the monotonicity property for a more general class of links. In fact, let* $\underline{L}$ *be a link on* Σ_g*, and* $B_1, ..., B_s$ *be the connected components of* $\Sigma_g \setminus \underline{L}$*. Let* k_i *be the number of boundary components of* $\overline{B}_i$ *(the closure of* B_i *in* Σ_g*),* g_i *be its genus, and* A_i *be its area. Assume the following is satisfied:*

- *for all* i*,* $\overline{B}_i$ *contains exactly one non contractible component of* $\underline{L}$ *around each of its handles (otherwise said, the non contractible components of the link generate a* g*-dimensional subspace of the first homology of* Σ_g*);*
- *if* $g_i > 0$*,* $k_i \geq 2$*;*
- *there exists constants* λ *and* η *such that for all* $1 \leq i \leq s$*,* $A_i + 2\eta(k_i + 2g_i - 1) = \lambda$*.*

Then, $\underline{L}$ *satisfies the monotonicity property: for all* $[u]$ *in the image of* $\pi_2(X, Sym(\underline{L})) \to H_2(X, Sym(\underline{L}))$*, one has* $\omega_X([u]) + \eta\Delta \cdot [u] = \frac{\lambda}{2}\mu([u])$*. As in [21], it is enough to check the monotonicity property on the basic disc classes tautologically corresponding to the* $\overline{B}_i$*'s. When they are planar (*$g_i = 0$*), it is shown in [21] that they satisfy the monotonicity property by embedding* $\overline{B}_i$ *into the sphere. If* $g_i \neq 0$*, we can embed* $\overline{B}_i$ *into* Σ_{g_i} *by gluing* k_i *discs of area* λ *on the boundary components. Then, we can apply the result of this section to get the monotonicity property for* $\overline{B}_i$*.*

2.3.2 Intersections in the symmetric product

We define elementary intersections between two strands: they are local descriptions for intersections counting towards $u \cdot \Delta$ which do not lift along the projection to the symmetric product. The goal of this section is to prove that elementary intersections are transverse in the symmetric product, and may thus be used to compute the intersection product with the diagonal.

First off, let us reduce the computation of the intersection product to counting the intersections between two strands.

Let $u : [0,1] \times [0,1] \to \mathrm{Sym}^k(\Sigma_{g,p})$ be a capping for an intersection point between the $\mathrm{Sym}^{k+g}\underline{L}$ and $\mathrm{Sym}^{k+g}\varphi(\mathrm{Sym}^{k+g}\underline{L})$. Assume that for some (s_0, t_0), $u(s_0, t_0) \in \Delta$. Generically, $u(s_0, t_0)$ belongs in the top stratum of the diagonal, which means that considering the path in the symmetric product

$$t \mapsto u(s_0, t)$$

as a collection of $k + g$ curves, at t_0 exactly two of them coincide (and there are no other intersections at t_0 or different t-we may assume this generically). Denote these two paths by $u_2 : [0,1] \times [0,1] \to \mathrm{Sym}^2\Sigma_g$, and the 2-dimensional diagonal by Δ_2.

Lemma 2.3.3 (Locality of the intersection problem)**.** *In the setting as above, the sign of the intersection at* (s_0, t_0) *between* u *and* Δ *is the same of the intersection at* (s_0, t_0) *between* u_2 *and* Δ_2*.*

Proof. Let us now consider a local chart of $\mathrm{Sym}^{k+g}(\Sigma_g)$ adapted to Δ at the intersection $u(s_0,t_0) = [x_1, x_1, x_3, \ldots, x_{k+g}] \in \Delta$. Consider the function $\mathrm{Sym}^2(\Sigma_g) \to \mathrm{Sym}^{k+g}(\Sigma_g)$ defined via

$$[y_1, y_2] \mapsto [y_1, y_2, x_3, \ldots, x_{k+g}] \tag{2.6}$$

This map is a complex embedding: it is clearly a continuous injective maps between compact topological spaces, hence a homeomorphism on its image, and its differential is injective. To prove this last point, we consider local charts given by symmetric polynomials. In formulas,

$$\mathrm{Sym}^2(\mathbb{C}) \cong \mathbb{C}^2, [y_1, y_2] \mapsto (y_1 + y_2, y_1 y_2) \tag{2.7}$$

$$\mathrm{Sym}^{k+g}(\mathbb{C}) \cong \mathbb{C}^{k+g}, [x_1, \ldots, x_{k+g}] \mapsto (e_1(x_1, \ldots, x_{k+g}), \ldots, e_{k+g}(x_1, \ldots, x_{k+g})) \tag{2.8}$$

In the equation above, e_i is the i-th symmetric polynomials on $k+g$ variables,

$$e_i(x_1, \ldots, x_{k+g}) := \sum_{1 \leq l_1 < \cdots < l_i \leq k+g} \prod_{j=1}^{i} x_{l_j} \tag{2.9}$$

In particular, e_1 and e_2 on two variables are the expressions appearing in (2.6). To prove injectivity of the differential, we just have to compute four derivatives. When restricting to the image of the topological embedding,

$$e_1(y_1, y_2, x_3, \ldots, x_k) = y_1 + y_2 + x_3 + \cdots + x_{k+g} \tag{2.10}$$

$$e_2(y_1, y_2, x_3, \ldots, x_{k+g}) = (y_1+y_2)(x_3+\cdots+x_{k+g})+y_1y_2+x_3x_4+\cdots x_{k+g-1}x_{k+g} \tag{2.11}$$

One can see that the topological embedding is in fact complex. Now, in local charts,

$$\partial_{y_1+y_2} e_1 = 1,\ \partial_{y_1 y_2} e_1 = 0,\ \partial_{y_1+y_2} e_2 = x_3 + \cdots + x_{k+g},\ \partial_{y_1 y_2} e_2 = 1 \tag{2.12}$$

that which proves injectivity of the differential. Now, we see $\mathrm{Sym}^2(\Sigma_g)$ as a complex submanifold of $\mathrm{Sym}^{k+g}(\Sigma_g)$, and remark that the embedding maps Δ_2 into Δ.

Assume now that the $k+g-2$ strands which do not intersect do not move on an interval $(s_0 - \varepsilon, s_0 + \varepsilon)$ (we do not lose generality since we are just reparametrising the capping). This condition implies that $Im(d_{(s_0,t_0)}u) \subset T_{(u(s_0,t_0))}\mathrm{Sym}^2(\Sigma_{g,p})$ (it is implicit in the notation that $u(s_0,t_0) \in \Delta_2$). Now, choose a complex chart around $u(s_0,t_0)$ adapted to $\mathrm{Sym}^2(\Sigma_{g,p})$: such a chart maps $\mathrm{Sym}^2(\Sigma_{g,p})$ locally to $\mathbb{C}^2 \times \{0\}^{k+g-2}$ and $T_{u(s_0,t_0)}\Delta$ also splits:

$$T_{u(s_0,t_0)}\Delta \cong V \oplus \mathbb{C}^{k+g-1}$$

where V is a complex line in $\mathbb{C}^2$ (we have $\mathbb{C}^{k+g-1}$ as second factor because the other $k+g-1$ coordinates may change as they wish, it won't change the fact that the point is in the diagonal because of the first two coordinates). Therefore, in order to compute the sign of the intersection, we are led to consider the determinant of the real matrix

$$M := \begin{pmatrix} v & w & x & y & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \tag{2.13}$$

which is a block matrix. Here $v, w \in \mathbb{R}^4$ are a real basis of $Im(d_{(s_0,t_0)}u)$, and $x, y \in \mathbb{R}^4$ are similarly a real basis of V. Clearly

$$\det(M) = \det(v, w, x, y) \tag{2.14}$$

which is what we wanted to show. $\square$

Definition 2.3.4. *Let $(s,t) \in [0,1]^2$. We say that a capping u has an elementary intersection at (s,t) with Δ if $u(s,t) \in \Delta$ and if there exist charts as those defined in Lemma 2.3.3 around $u(s,t)$ such that u_2 coincides with the roots of the complex polynomial*

$$X^2 - (s + it)$$

Lemma 2.3.5. *Elementary intersections between two strands are transverse.*

Proof. We will start by considering a homotopy $\gamma : [-1,1]_s \times [-1,1]_t \to \mathrm{Sym}^2(\mathbb{C})$ between two braids γ_- and γ_+. There is a diffeomorphism $\varphi : \mathbb{C}^2 \to \mathrm{Sym}^2(\mathbb{C})$ which maps $(a,b) \in \mathbb{C}^2$ to the pair of roots of the degree 2 polynomial $X^2 - aX + b$ (its inverse being given by $\varphi^{-1}([x_-, x_+]) = (x_- + x_+, x_- x_+)$). Through this diffeomorphism, the diagonal $\Delta \subset \mathrm{Sym}^2(\mathbb{C})$ corresponds to the set

$$\varphi^{-1}(\Delta) = \left\{(2x, x^2), x \in \mathbb{C}\right\} = \left\{\left(a, \frac{a^2}{4}\right), a \in \mathbb{C}\right\}$$

whose tangent space at $(a, \frac{a^2}{4})$ is the complex vector subspace generated by $(1, \frac{a}{2})$, i.e. the real vector subspace generated by $(1, \frac{a}{2})$ and $(i, \frac{ia}{2})$.

Let $(a(s,t), b(s,t)) := \varphi^{-1}(\gamma(s,t))$. Assume that γ intersects the diagonal at $(s,t) = (0,0)$. Then, the intersection is transverse if and only if the vectors

$$\left(1, \frac{a(0,0)}{2}\right), \left(i, \frac{ia(0,0)}{2}\right), (\partial_s a(0,0), \partial_s b(0,0)) \text{ and } (\partial_t a(0,0), \partial_t b(0,0))$$

generate the whole space $\mathbb{C}^2$ as a real vector space.

In particular, consider the case where $\gamma(s,t)$ is the pair of square roots of $s+it$. Then, γ intersects the diagonal at $(0,0)$, and $(a(s,t), b(s,t)) = (0, -s-it)$. Therefore, we get that the intersection is transverse if and only if the vectors $(1,0), (i,0), (0,-1), (0,-i)$ generate $\mathbb{C}^2$. Since this is true, we have transversality of the intersection. $\square$

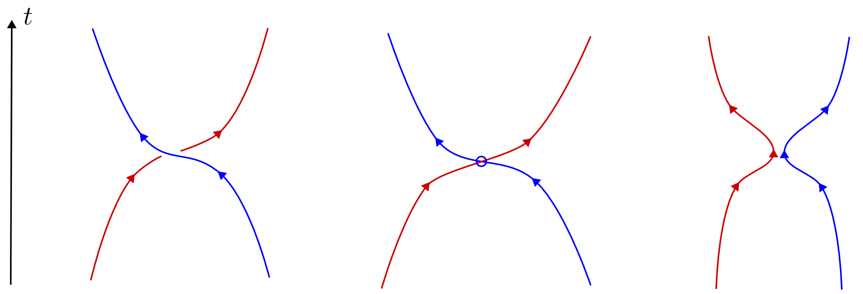


Figure 2.3: The homotopy $\gamma(s, t)$

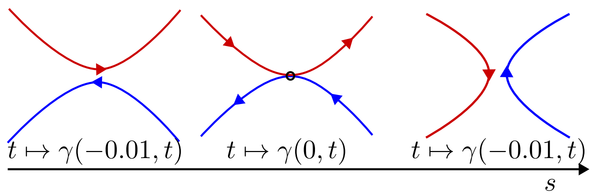


Figure 2.4: The homotopy $\gamma(s, t)$ as seen on $\mathbb{C}$.

Remark 2.3.6. *From the proof we see that the sign of the intersection we studied above as local model is positive.*

Remark 2.3.7. *The homotopy $s + it \mapsto \sqrt{(s + it)} \in Sym^2\mathbb{C}$ does not lift to a function to $\mathbb{C}^2$. If we, in more generality, suppose that an intersection is transverse, the capping cannot be lifted to $\mathbb{C}^2$. Assume in fact that an intersection counting towards $[u] \cdot \Delta$ does lift to $\mathbb{C}^2$ (we take $k = 2$ in light of Lemma 2.3.3). Then it cannot be transverse: up to time-translation, assume the intersection appears at $(s, t) = (0, 0)$. Define*

$$(\gamma_1, \gamma_2) : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \mathbb{C}^2$$

such that if $\pi : \mathbb{C}^2 \to Sym^2\mathbb{C}$ is the quotient projection, we locally have

$$\pi \circ (\gamma_1, \gamma_2) = u$$

Then $(\gamma_1, \gamma_2)(0, 0) \in \pi^{-1}(\Delta)$. The differential of π being 0 there, so is the differential of u. This proves that the intersection of u with Δ at $(0, 0)$ is not transverse.

2.3.3 The braid type function

In this Section we discuss the good definition and the image of the braid-type function b.

In the introduction it was claimed that b was a well defined function from $\mathrm{Ham}_c(\Sigma_{g,p})$. To prove it, we need to remark that b does not depend on the Hamiltonian isotopy one chooses. This is true if, for instance, $\pi_1(\mathrm{Ham}_c(\Sigma_{g,p})) = 0$, which we proved in Lemma 1.2.8. In fact, since in such a case any two Hamiltonian isotopies are homotopic relative endpoints between isotopies, we may deform one Hamiltonian path into the other keeping strands from crossing. Such a deformation then provides a braid isotopy between the images of the two Hamiltonian isotopies, and b is well defined.

We aim to describe the braid type of a Hamiltonian diffeomorphism preserving a given premonotone Lagrangian configuration $\underline{L}$ in $\Sigma_{g,p}$. We refer to the description of $\mathcal{B}_{k,g,p}$ provided in Section 1.1. Consider the subgroup of $\mathcal{B}_{k,g,p}$ generated by $(\sigma_i)_{i=1,\dots,k-1}$, $(a_j)_{j=1,\dots,g}$, $(b_j^{-1}a_jb_j)_{j=1,\dots,g}$ and $(z_l)_{l=1,\dots,p-1}$ only (no b_j alone is present), with the restrictions of Bellingeri's relations. For clarity, we remark that the σ_i we consider are those describing exchanges between contractible components of the link. Denote this subgroup by $\mathcal{B}_{\underline{L}}$.

Remark 2.3.8. *Even though we have p punctures, the associated generators are only $p-1$. As remarked in [10], one may express a single loop z_p around the last puncture using the relation*

$$[a_1, b_1^{-1}]\cdots[a_g, b_g^{-1}] = \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{k-1}^{-2}\cdots\sigma_2^{-1}\sigma_1^{-1}z_1^{-1}\cdots z_p^{-1}$$

Lemma 2.3.9. *Let us consider the group homomorphism*

$$b : Ham_{\underline{L},c}(\Sigma_g) \to \mathcal{B}_{k,g,p} \tag{2.15}$$

Its image is precisely $\mathcal{B}_{\underline{L}}$.

Proof. We are going to describe the group $\mathcal{B}_{\underline{L}}$ as the fundamental group of the k-th configuration space of the surface $\tilde{\Sigma}_{g,p}$, which we obtain by removing tubular neighbourhoods of the curves $L_{k+1},\dots,L_{k+g}$ from $\Sigma_{g,p}$. Now, $\tilde{\Sigma}_{g,p}$ is a punctured sphere, with $2g+p$ punctures. Its braid group is described in [11, Theorem 2.1]: it has the $k-1$ generators corresponding to moves which take place in a disc on the surface, and $p+2g-1$ corresponding to non contractible loops around punctures.

Let us show that we may identify the image of b with this fundamental group. Let $\varphi \in \mathrm{Ham}_{\underline{L},c}(\Sigma_{g,p})$, and let L_j be any non contractible component of $\underline{L}$. Since it cannot be displaced by any Hamiltonian diffeomorphism, if (φ^t) is any homotopy between the identity of Ham and φ, L_j satisfies

$$L_j \cap \varphi^t(L_j) \neq \emptyset$$

This implies that there is a loop in $\mathrm{Ham}_c(\Sigma_{g,p})$ based at the identity, $t \mapsto \psi^t$, such that for all t and for any non contractible component L_j one has

$$\psi^t\varphi^t(L_j) = L_j$$

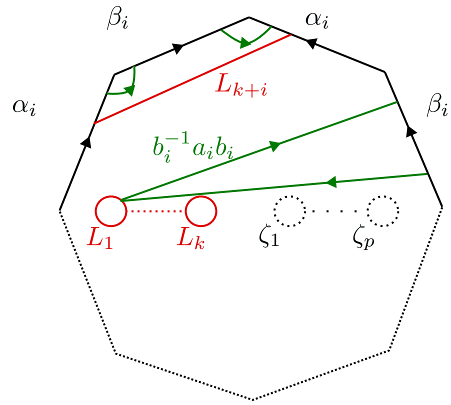


Figure 2.5: The generator $b_i^{-1}a_ib_i$

Since the circles cannot intersect during this isotopy, and since b does not depend on the particular isotopy one chooses without loss of generality (up to shrinking the tubular neighbourhoods we take off to produce $\tilde{\Sigma}_{g,p}$) the braid $b(\varphi)$ is represented by a braid entirely contained in $\tilde{\Sigma}_{g,p}$. Furthermore, choosing such an isotopy makes it clear that no contractible circle may wind about the base point on a non contractible component: this bounds the number of σ-generators to $k-1$ (only exchanges between contractible circles are allowed). All of this proves that the image of b is contained in $\pi_1(\mathrm{Conf}^k(\tilde{\Sigma}_{g,p}))$, which is what we wanted. To prove equality, we just need to be able to represent all generators of $\mathcal{B}_{\underline{L},k}$ as images of b. This is readily done: given any σ_i, we may consider a disc containing L_1 and L_i and apply a half rotation. In the case of any z_i (loop around a puncture) we may use an annulus with centre at the puncture, and a full rotation on it.

We end this proof showing that the generators above, elements in $\mathcal{B}_{k,g,p}$, are in fact the images of the generators of $\pi_1(\mathrm{Conf}^k(\tilde{\Sigma}_{g,p}))$ under the inclusion morphism. In fact, it is easy to see that the σ_i's in $\mathcal{B}_{k,g,p}$ correspond to the half-twists in $\pi_1(\mathrm{Conf}^k(\tilde{\Sigma}_{g,p}))$, and that the turns around the punctures in $\tilde{\Sigma}_{g,p}$ are sent by the inclusion to turns around the punctures in $\Sigma_{g,p}$ (the z_j's) and turns around the handles (the a_i's and $b_i^{-1}a_ib_i$'s) (see Figure 2.5 for a picture of $b_i^{-1}a_ib_i$).

□

2.4 Proof of the main results

As the title says, in this Section we prove the main results of the Chapter. A sketch of the proof is the following: consider $\varphi \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$, with $\underline{L}$ pre-monotone. We symplectically embed the surface with boundary $\Sigma_{g,p}$ into two closed surfaces, gluing discs of different areas to the boundary components. This operation yields two different Floer complexes, which turn out to be chain-isomorphic; the isomorphism however does not respect the action filtration. We compute the action difference, and relate it to the braid type of the Hamiltonian diffeomorphism φ. In Section 2.4.1 we construct the chain isomorphism and study its effects on spectral invariants, while in Section 2.3.2 we finish the proof connecting the difference of spectral invariants to braids. This difference of spectral invariants will be what we use in the definition of the functions $\mathfrak{f}_{(v_1,v_2)}$ in Theorem 2.2.3, and the quasimorphisms in Theorem 2.2.6.

2.4.1 Construction of the chain isomorphism

The content of this section comes from [51], but it is adapted to the more general setting of surfaces with boundary and arbitrary genus.

Let $\underline{L}$ be a pre-monotone link on $\Sigma_{g,p}$ with $k+g$ components, and let $\zeta_1, ..., \zeta_p$ be its boundary components. Gluing discs of area $s_{i,j}$ along ζ_j for $i = 1, 2$, $j = 1, ..., p$ gives rise to two embeddings of $\Sigma_{g,p}$ into a closed surface of area $1 + s_i$, $i = 1, 2$, denoted $\Sigma_g(1 + s_i)$, where

$$s_i = s_{i,1} + ... + s_{i,p} \in (0, (k+1)A - 1)$$

In the following, by v_i we denote the vector

$$v_i := (s_{i,1}, \ldots, s_{i,p})$$

and the set of possible values v_1, v_2 may take is

$$V := \{(s_1, ..., s_p) \in (\mathbb{R}_{\geq 0})^p | s_1 + ... + s_p \leq (k+1)A - 1\}$$

We write $j_v : \Sigma_{g,p} \to \Sigma_g(1 + s)$ for the symplectic embedding we obtain as just described. For any $\varphi \in \mathrm{Ham}_c(\Sigma_{g,p})$ denote by $\varphi_v \in \mathrm{Ham}(\Sigma_g(1 + s))$ the obvious extension by the identity, $\underline{L}_v := j_v(\underline{L})$ the corresponding monotone link on $\Sigma_g(1+s)$, and by η_s the associated monotonicity constant. If φ is generated by a Hamiltonian $H \in \mathscr{C}_c^\infty(\Sigma_{g,p} \times \mathbb{S}^1; \mathbb{R})$, let H_v be the Hamiltonian generating φ_v coinciding with H on the image of j_v. In this Section we assume H to be non degenerate: it cannot be the case of course if $\varphi \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p}, \omega)$, so later we shall have to consider small perturbations to make the intersections transverse.

Fix a diffeomorphism of the surface

$$d_{v_1}^{v_2} : \Sigma_g(1 + v_1) \to \Sigma_g(1 + v_2)$$

and assume that for all j, $v_{1,j}, v_{2,j} > 0$. We require that the following diagram commute:

$$\begin{array}{ccc} \Sigma_g(1+s_1) & \xrightarrow{d_{v_1}^{v_2}} & \Sigma_g(1+s_2) \\ {\scriptstyle j_{v_1}}\uparrow & \nearrow{\scriptstyle j_{v_2}} & \\ \mathbb{D} & & \end{array}$$

In particular, since j_{v_i} is symplectic for $i = 1, 2$, by the commutativity of the above diagram we have that $d_{v_1}^{v_2}$ is symplectic between the images of $\mathbb{D}$ in the two different spheres and globally preserves the orientations given by the symplectic forms. The map $d_{v_1}^{v_2}$ induces a bijective correspondence between the Hamiltonian paths[1] which, after being capped, generate the Floer complexes $CF(H_{v_1}, \underline{L}_{v_1})$ and $CF(H_{v_2}, \underline{L}_{v_2})$ for any non degenerate $\varphi \in \mathrm{Ham}_c(\Sigma_{g,p})$. We would like to use $\psi_{v_1}^{v_2} := \mathrm{Sym}^k(d_{v_1}^{v_2})$ to define a chain-isomorphism, and the first step to do so is checking that $\psi_{v_1}^{v_2}$ commutes with the $\mathbb{Z}$-action on both sides given by recapping. Before stating the lemma, recall that $\psi_{v_1}^{v_2}$ is smooth because it is induced by a biholomorphism $\Sigma_g(1+s_1) \to \Sigma_g(1+s_2)$, hence biholomorphic itself (after fixing the complex structure on $\Sigma_g(1+s_1)$, the other one is its pushforward by $d_{v_1}^{v_2}$, and it is integrable by Newlander–Nirenberg Theorem [48, Appendix E]). Two ways to see it for instance are applying the Removable Singularity Theorem on the diagonal (where the function may not be holomorphic), or checking by hand using Cauchy's Integral Formula that, even when a point of $\mathrm{Sym}^k(\Sigma_g)$ has non trivial isotropy, $\psi_{v_1}^{v_2}$ may be developed in a power series using the holomorphic coordinates of the symmetric product.

We prove now that $\psi_{v_1}^{v_2}$ determines a bijection $\Psi_{v_1}^{v_2}$ between capping classes.

Lemma 2.4.1. *If $\hat{y}$, $\hat{y}'$ are two equivalent cappings for an intersection point y, then $\psi_{v_1}^{v_2}(\hat{y})$, $\psi_{v_1}^{v_2}(\hat{y}')$ are two equivalent cappings for the intersection point $\psi_{v_1}^{v_2}(y)$. Moreover, the associated $\mathbb{Z}$-action induces a translation by multiples of λ of the action for both $CF(H_{v_1}, \underline{L}_{v_1})$ and $CF(H_{v_2}, \underline{L}_{v_2})$.*

Proof. Start by remarking that $\psi_{v_1}^{v_2}$ induces an identification between the relative homology groups: if X_{v_i} denotes the quotient $\Sigma_g(1+s_i)^k/\mathfrak{S}_k$ as above, and $L_{v_i} := \mathrm{Sym}^k(j_{v_1})(\mathrm{Sym}^k(\underline{L}))$ $i = 1, 2$, then

$$\psi_{v_1 *}^{v_2} : H_2(X_{v_1}, L_{v_1}; \mathbb{Z}) \xrightarrow{\sim} H_2(X_{s_2}, L_{v_2}; \mathbb{Z})$$

The restriction of $\psi_{v_1}^{v_2}$ to the image of the Hurewicz morphism $H_2^D(X_{v_1}, L_{v_1}; \mathbb{Z}) \leq H_2(X_{v_1}, L_{v_1}; \mathbb{Z})$ is still an isomorphism. To check that the equivalence classes of cappings are respected it suffices to check that given two classes u, u' in $H_2^D(X_{v_1}, L_{v_1}; \mathbb{Z})$ satisfying

$$\langle \omega_{X_{v_i}}, u \rangle + \eta_{v_1}[u] \cdot \Delta_{s_1} = \langle \omega_{X_{v_i}}, u' \rangle + \eta_{v_1}[u'] \cdot \Delta_{s_1}$$

then

$$\langle \omega_{X_{s_2}}, \psi_{v_1 *}^{v_2} u \rangle + \eta_{v_2}[\psi_{v_1 *}^{v_2} u] \cdot \Delta_{s_2} = \langle \omega_{X_{s_2}}, \psi_{v_1 *}^{v_2} u' \rangle + \eta_{v_2}[\psi_{v_1 *}^{v_2} u'] \cdot \Delta_{s_2}$$

Now, Lemma 4.19 from [21] shows that

$$\langle \omega_{X_{v_i}}, u \rangle + \eta_{v_1}[u] \cdot \Delta_{s_1} = \frac{\lambda}{2} \mu(u)$$

for each $u \in H_2^D(X_{v_1}, L_{v_1}; \mathbb{Z})$ where $\mu \in H^2(X_{v_1}, L_{v_1}; \mathbb{Z})$ is the Maslov class; an analogous statement holds for classes in $H_2^D(X_{s_2}, L_{v_2}; \mathbb{Z})$. In light of this, $\hat{y}$

[1] This is due to the commutativity of the diagram above and the fact that the extensions of the Hamiltonian are 0 outside small neighbourhoods of the images $j_{v_i}(\Sigma_{g,p})$.

and $\hat{y}'$ are equivalent cappings if and only if their images through the Hurewicz morphism, denoted u and u' respectively, satisfy

$$\mu(u) = \mu(u')$$

An analogous result is true for $\psi_{v_1}^{v_2} u$ and $\psi_{v_1}^{v_2} u'$. To prove the first statement it is now enough to show that

$$\mu(u) = \mu(\psi_{v_1 *}^{v_2} u)$$

and the analogous conclusion for u'. These are true because $\psi_{v_1}^{v_2}$ is a biholomorphism, and thus preserves the Maslov indices (see [49], Theorem C.3.7).

To prove the second statement, we need to prove that the generators of $H_D(X_{v_i}, L_{v_i}; \mathbb{Z})$ act the same way on the action for $i = 1, 2$. Notice that in both cases $\lambda_1 = \lambda_2 = A =: \lambda$ the area of a disk bounded by a component of the lagrangian link. Now, capping by one of the generators of $H_D(X_{v_i}, L_{v_i}; \mathbb{Z})$ shifts the action by $\lambda = A$ in both cases since the generators have Maslov index 2, see Corollary 4.8 and Lemma 4.19 in [21]. □

Applying the above Lemma to $\psi_{v_2}^{v_1} = (\psi_{v_1}^{v_2})^{-1}$ we prove that the map$\Psi_{v_1}^{v_2}$ defined on the generators of the complexes by

$$[\hat{y}] \mapsto [\psi_{v_1}^{v_2} \circ \hat{y}]$$

is a bijection between capping classes which extends linearly to a chain complex isomorphism.

Fix an ω-tame almost complex structure on $\Sigma_g(1 + s_1)$ and push it forward by $\psi_{v_1}^{v_2}$. Let H be a Hamiltonian generating φ, and $\psi_{v_1}^{v_2}$ be the group morphism induced by $\psi_{v_1}^{v_2}$.

Lemma 2.4.2. *Let $\varphi : \Sigma_{g,p} \to \Sigma_{g,p}$ be a Hamiltonian diffeomorphism of the symplectic surface such that $\varphi(\underline{L}) \pitchfork \underline{L}$. Then $\psi_{v_1}^{v_2}$ is a chain isomorphism, and in particular $HF(H_{v_1}, \underline{L}_{v_1}) \simeq HF(H_{v_2}, \underline{L}_{v_2})$.*

Proof. Let us denote by ∂^{v_i}, $i = 1, 2$, the differentials of the two complexes: we want to prove that

$$\partial^{v_2} \circ \psi_{v_1}^{v_2} = \psi_{v_1}^{v_2} \circ \partial^{v_1}$$

To achieve the transversality in the symmetric product one needs to define the Floer complex we look at a class of almost complex structures which coincide with the one induced by the quotient projection near the diagonal: for $i = 1, 2$ let

$$\mathcal{J}_{v_i}(\Delta)$$

be the set of almost complex structures on $\mathrm{Sym}^k(\Sigma_g(1+s_i))$ which coincide with $J_{X_{v_i}}^{s_i}$ on a neighbourhood of the fat diagonal $\Delta \subset X_{v_i}$, and elsewhere they are tamed by $\omega_{X_{v_i}}$. Here we use $\omega_{X_{v_i}}$ and $J_{X_{v_i}}^{s_i}$ for the natural (singular) symplectic form and almost complex structure on the quotient of the remarks above.

Let now $u : \mathbb{R} \times \mathbb{S}^1 \to \mathrm{Sym}^k(\Sigma_g(1+s_1))$ be a smooth function satisfying the conditions

$$\begin{cases} u(s,0) \in \mathrm{Sym}^k(\varphi_{v_1})(\mathrm{Sym}^k(\underline{L})) \\ u(s,1) \in \mathrm{Sym}^k(j_{v_1})(\mathrm{Sym}^k(\underline{L})) \\ \lim_{s\to-\infty} u(s,t) = j_{v_1} y_0 \\ \lim_{s\to+\infty} u(s,t) = j_{v_1} y_1 \\ \partial_s u(s,t) + J_t^{v_1}(\partial_t u(s,t) - X_{H_{v_1}} u(s,t)) = 0 \end{cases} \tag{2.16}$$

where $J^{v_1} \in \mathcal{J}_{v_1}(\Delta)$ is an almost complex structure on $\Sigma_g(1+s_1)$. Endow $\Sigma_g(1+s_2)$ with the almost complex structure $J^{v_2} := \psi^{v_2}_{v_1 *} J^{v_1}$ (the push-forward by $\psi^{v_2}_{v_1}$ of J^{v_1}), so that by definition the differential of $\psi^{v_2}_{v_1}$ is complex linear.

We want then to show that J^{v_2} thus defined is an element of $\mathcal{J}_{v_2}(\Delta)$: we only need to show that J^{v_2} is tamed by $\omega_{X_{s_2}}$ if J^{v_1} is tamed by $\omega_{X_{v_i}}$. Tameness being an open condition, and since the natural complex structure is tamed by the natural symplectic structure, let us assume for the moment that J^{v_1} is close enough to $J^{v_1}_{X_{v_i}}$. The pushforward $\psi^{v_2}_{v_1 *}$ defines a homeomorphism between the spaces of almost complex structures on X_{v_i} and X_{s_2}, and sends by definition $J^{v_1}_{X_{v_i}}$ to $J^{v_2}_{X_{s_2}}$: this implies that if J^{v_1} is close enough to $J^{v_1}_{X_{s_2}}$ then $J^{v_2} := (\psi^{v_2}_{v_1})_* J^{v_1}$ is close to $J^{v_2}_{X_{s_2}}$ and is tame: $J^{v_2} \in \mathcal{J}_{v_2}(\Delta)$.

A bijective correspondence between holomorphic curves $u_1 : \mathbb{D} \to X_{v_i}$ and $u_2 : \mathbb{D} \to X_{s_2}$ is given by the post-composition with $\psi^{v_2}_{v_1}$ (or its inverse), since $\psi^{v_2}_{v_1}$ conjugates the Hamiltonian vector fields on X_{v_1} and X_{v_2}. We are left to show that the differentials are defined at the same time, i.e. that if J^{v_1} achieves the transversality one needs for the good definition of the Floer complex, then so does J^{v_2}. As mentioned above we have a well defined homeomorphism

$$(\psi^{v_2}_{v_1})_* : \mathcal{U}^1 \xrightarrow{\sim} \mathcal{U}^2$$

where $\mathcal{U}^i \subset \mathcal{J}_{v_i}(\Delta)$ are neighbourhoods of the natural almost complex structures; the inverse is given by the pullback via $\psi^{v_2}_{v_1}$. If $\mathcal{J}_{v_i,\tau}(\Delta) \subset \mathcal{J}_{v_i}(\Delta)$ is the Baire set of almost complex structures giving good definition for the Floer complex (proof in [21], Section 5), let us prove that $\mathcal{J}_{v_1,\tau}(\Delta) \cap (\psi^{v_2}_{v_1})^*(\mathcal{U}^2 \cap \mathcal{J}_{v_2,\tau}(\Delta)) \neq \emptyset$. If it is the case, we may choose J^{v_1} in it, and push it forward to J^{v_2}, so both Floer complexes can be defined with these choices. Now, the intersection cannot be empty: since $\mathcal{J}_{v_2,\tau}(\Delta)$ is generic in $\mathcal{J}_{v_2}(\Delta)$, $\mathcal{U}^2 \cap \mathcal{J}_{v_2,\tau}(\Delta)$ is a Baire set in $\mathcal{U}^2$, and $(\psi^{v_2}_{v_1})^*(\mathcal{U}^2 \cap \mathcal{J}_{v_2,\tau}(\Delta))$ is Baire in $\mathcal{U}^1$. Intersecting the latter with $\mathcal{J}_{v_1,\tau}(\Delta)$ gives a Baire set in $\mathcal{U}^1$, which is in particular not empty.

As for the orientation of moduli spaces, one can arbitrarily define a spin structure on $\mathrm{Sym}^k(\underline{L})$ already on the disk, that which gives spin structures on on $\mathrm{Sym}^k(\underline{L}_{v_2})$ and $\mathrm{Sym}^k(\underline{L}_{v_1})$ by pushforward by j_{v_i}, and then these two correspond by pushforward by $\psi^{v_2}_{v_1}$ since $\psi^{v_2}_{v_1}$ preserves the orientation of the Lagrangian link. □

Remark 2.4.3. *The isomorphism between the Floer complexes in Lemma 2.4.2 is an isomorphism of persistence modules only up to shift: what we are going to do later essentially amounts to computing how much it fails to preserve the action filtration.*

Let $Q\psi_{v_1}^{v_2} : QH(\mathrm{Sym}^k(\underline{L}_{v_1});\mathbb{Z}) \to QH(\mathrm{Sym}^k(\underline{L}_{v_2});\mathbb{Z})$ be the morphism on quantum homology induced by $\psi_{v_1}^{v_2}$.

Lemma 2.4.4. *Let $PSS(v_i) : QH(Sym^k(\underline{L}_{v_i});\mathbb{Z}) \to HF(\varphi_{s_i}, \underline{L}_{v_i};\mathbb{Z})$ be the PSS isomorphisms for $i = 1, 2$. Then the following diagram commutes:*

$$\begin{array}{ccc} QH(Sym^k(\underline{L}_{v_1});\mathbb{Z}) & \xrightarrow{Q\psi_{v_1}^{v_2}} & QH(Sym^k(\underline{L}_{v_2});\mathbb{Z}) \\ \downarrow{\scriptstyle PSS(v_1)} & & \downarrow{\scriptstyle PSS(v_2)} \\ HF(H_{v_1}, \underline{L}_{v_1};\mathbb{Z}) & \xrightarrow{\psi_{v_1}^{v_2}} & HF(H_{v_2}, \underline{L}_{v_2};\mathbb{Z}) \end{array}$$

Proof. The matters of definitions of the PSS isomorphism and its bijectivity in the case of symmetric products are addressed in [21], Section 6.2: modifying the (singular) Hamiltonian $\mathrm{Sym}^k(H)$ near the diagonal in fact one can find a filtered chain isomorphic complex for which the standard proofs work. In the following we assume that this operation has been done.

As in the proof for $\psi_{v_1}^{v_2}$, we need to check that $\psi_{v_1}^{v_2}$ induces a bijective correspondence of generators of the chain complexes and of moduli spaces defining the differential. This is immediately clear for $Q\psi_{v_1}^{v_2}$: gradient lines on $\mathrm{Sym}^k(\underline{L}_{v_1})$ are mapped to gradient lines if one fixes a Morse-Smale pair and then pushes it forward, and to holomorphic discs in the pearly model for $QC(\mathrm{Sym}^k(\underline{L}_{v_1}))$ we make bijectively correspond holomorphic discs in the pearly model for $QC(\mathrm{Sym}^k(\underline{L}_{v_2}))$. As for the orientation of the involved moduli spaces, the same remark as above holds. □

2.4.2 Filtration shifts and braids

What follows is material from the joint work with Trifa [52].

We now study the action shift given by the chain isomorphism $\Psi_{v_1}^{v_2}$, and relate it to the braid type of the Hamiltonian diffeomorphism. We assume that φ generated by H is in fact a small perturbation of an element in $\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p}, \omega)$. Its braid type is in particular still well defined.

Lemma 2.4.5. *The difference of action $\mathcal{A}_{H_{v_1}}([\hat{y}]) - \mathcal{A}_{H_{v_2}}([\Psi_{v_1}^{v_2}\hat{y}])$ does not depend on the choice of the generator $[\hat{y}]$.*

Proof. Start by fixing a generator $[\hat{y}]$. Since H_{v_i}, $i = 1, 2$ are supported in $\Sigma_{g,p}$, we get that y_i are paths in $\mathrm{Sym}^{k+g}(\Sigma_{g,p})$. The biholomorphism between $\Sigma_g(1 + s_1)$ and $\Sigma_g(1 + s_2)$ restricts to the identity on $\Sigma_{g,p}$, so the two paths y and $\psi_{v_1}^{v_2} y$ coincide, and $H_{v_1}|_y = H_{v_2}|_{\psi_{v_1}^{v_2} y} = H|_y$. Therefore, using the formula for the action, the terms containing the integral of the Hamiltonian will cancel out, leaving us with:

$$\mathcal{A}_{H_{v_1}}([\hat{y}]) - \mathcal{A}_{H_{v_2}}([\Psi_{v_1}^{v_2}\hat{y}]) = \eta_{v_2}[\Psi_{v_1}^{v_2}\hat{y}] \cdot \Delta - \eta_{v_1}[\hat{y}] \cdot \Delta + \omega_{v_2}([\Psi_{v_1}^{v_2}\hat{y}]) - \omega_{v_1}([\hat{y}])$$

We first show that this quantity does not depend on the choice of capping for the path y. Let $[\hat{y}']$ be another capping. We want to show that

$$\eta_{v_2}([\Psi^{v_2}_{v_1}\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1})\cdot\Delta+\omega_{v_2}([\Psi^{v_2}_{v_1}\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1})- \\ -\left(\eta_{v_1}([\hat{y}]\#[\hat{y}']^{-1})\cdot\Delta+\omega_{v_1}([\hat{y}]\#[\hat{y}']^{-1})\right)=0$$

Since $[\hat{y}]\#[\hat{y}']^{-1}$ and $\Psi^{v_2}_{v_1}[\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1}$ are homotopies between x and itself, it is in fact a disc with boundary on $\mathrm{Sym}\underline{L}$, so by monotonicity (Proposition 2.3.1),

$$\eta_{v_1}([\hat{y}]\#[\hat{y}']^{-1})\cdot\Delta+\omega_{v_1}([\hat{y}]\#[\hat{y}']^{-1})=\frac{\lambda}{2}\mu([\hat{y}]\#[\hat{y}']^{-1})$$

$$\eta_{v_2}(\Psi^{v_2}_{v_1}[\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1})\cdot\Delta+\omega_{v_2}(\Psi^{v_2}_{v_1}[\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1})=\frac{\lambda}{2}\mu(\Psi^{v_2}_{v_1}[\hat{y}]\#\Psi^{v_2}_{v_1}[\hat{y}']^{-1})$$

Since the Maslov index is preserved by $\Psi^{v_2}_{v_1}$, the difference above vanishes, which proves that the difference of action does not depend on the choice of capping. Now we show that it does not depend on the path y either. If y' is another trajectory of H_{v_1} between $\mathrm{Sym}\underline{L}_{v_1}$ and itself, then we can define a capping for y' by choosing any capping for y and concatenating it with any homotopy $[w]$ from y to y'. Since we showed that the action difference does not depend on the capping, it is enough to find a single homotopy $[w]$ for which

$$\eta_{v_2}([\psi^{v_2}_{v_1}w])\cdot\Delta+\omega_{v_2}([\psi^{v_2}_{v_1}w])-(\eta_{v_1}([w])\cdot\Delta+\omega_{v_1}([w]))=0$$

Such a w exists: we can first find a homotopy sliding the end points of y and y' inside the circles so that they coincide (and therefore define braids), and then we can take a homotopy inside $\mathrm{Sym}^{k+g}(\Sigma_{g,p})$ that does not intersect Δ, because y and y' have the same braid type. □

The previous Lemma implies that for any $\varphi\in\mathrm{Ham}_{\underline{L},c}(\Sigma_{g,p})$, and any choice of generator $[\hat{y}]$, we have $c_{\underline{L}_{v_1}}(\varphi)-c_{\underline{L}_{v_2}}(\varphi)=\frac{1}{k+g}(\mathcal{A}_{H_{v_1}}([\hat{y}])-\mathcal{A}_{H_{v_2}}([\Psi^{v_2}_{v_1}\hat{y}]))$ as in [51, Lemma 3.6].

Definition 2.4.6. *We define $f_{v_1,v_2}:Ham_c(\Sigma_{g,p})\to\mathbb{R}$ as*

$$f_{v_1,v_2}(\varphi):=c_{\underline{L}_{v_1}}(\varphi)-c_{\underline{L}_{v_2}}(\varphi)=\frac{1}{k+g}(\mathcal{A}_{H_{v_1}}([\hat{y}])-\mathcal{A}_{H_{v_2}}(\Psi^{v_2}_{v_1}[\hat{y}]))$$

Moreover, we have:

Lemma 2.4.7. *The map $f_{v_1,v_2}:Ham_{\underline{L},c}(\Sigma_{g,p})\to\mathbb{R}$ is a group homomorphism, which factorises over $b:Ham_{\underline{L},c}(\Sigma_{g,p})\to\mathcal{B}_{\underline{L}}$, i.e. there exists a group homomorphism $\mathfrak{f}_{v_1,v_2}:\mathcal{B}_{\underline{L}}\to\mathbb{R}$ such that $f_{v_1,v_2}=\mathfrak{f}_{v_1,v_2}\circ b$.*

Proof. Let φ, ψ be in $\mathrm{Ham}_{\underline{L},c}(\Sigma_{g,p})$. Let H (resp. H') be a Hamiltonian supported inside $\Sigma_{g,p}$ generating φ (resp. ψ). Let $[\hat{y}]$ (resp. $[\hat{y'}]$) be a generator of $CF^*(\mathrm{Sym}^{k+g}H_{v_1},\mathrm{Sym}\underline{L}_{v_1})$ (resp. $CF^*(\mathrm{Sym}^{k+g}H'_{v_1},\mathrm{Sym}\underline{L}_{v_1})$. Let $z:[0,1]\to\mathrm{Sym}^{k+g}\Sigma_{g,p}$ be a trajectory of $H\#H'$ from $\mathrm{Sym}\underline{L}$ to itself. We want to construct a capping $\hat{z}:[0,1]\times[0,1]\to\mathrm{Sym}^{k+g}\Sigma_g(1+s_1)$ from the constant path x to z, using the cappings $\hat{y}$ and $\hat{y'}$. We define it as follows (see Fig. 2.6):

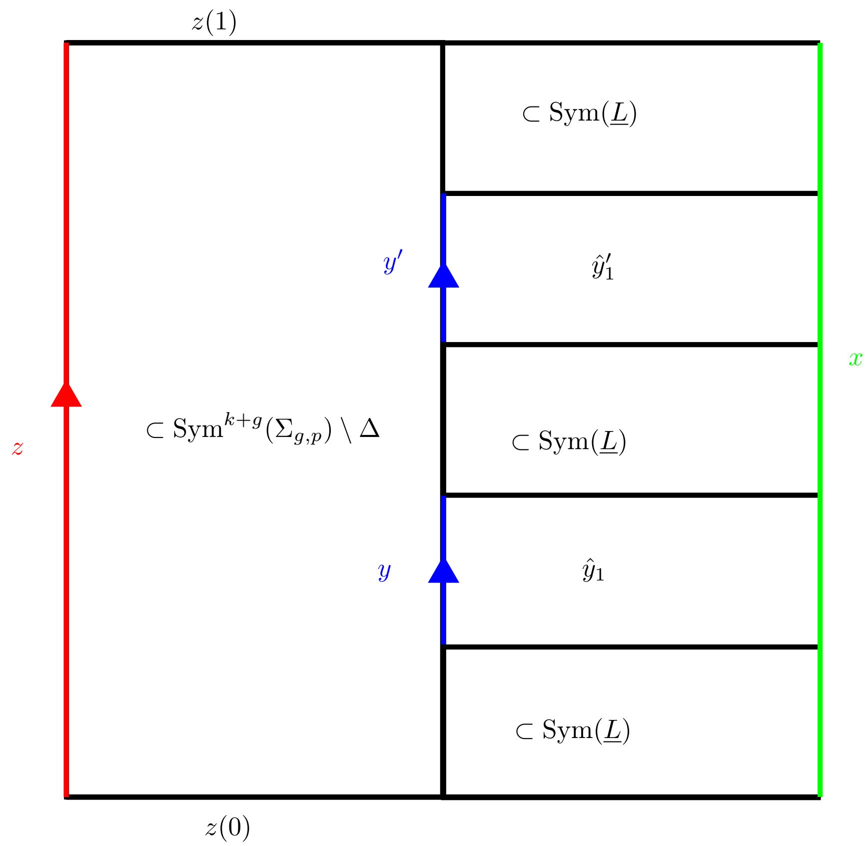


Figure 2.6: The capping $\hat{z}_1$

- On $[0, \frac{1}{2}] \times [0, \frac{1}{5}]$:
  - $\hat{z}([0, \frac{1}{2}] \times [0, \frac{1}{5}]) \subset \mathrm{Sym}\underline{L}_{v_1}$
  - for $0 \leq t \leq \frac{1}{5}$, $\hat{z}(0,t) = x$
  - $\hat{z}(\frac{1}{2}, 0) = z(0)$
  - for $0 \leq s \leq \frac{1}{2}$, $\hat{z}(s, \frac{1}{5}) = \hat{y}(2s, 0)$

  (Such a homotopy exists because $\mathrm{Sym}\underline{L}_{v_1}$ is path-connected, it is in fact isotopic to a Clifford torus, cf. [21])
- On $[0, \frac{1}{2}] \times [\frac{1}{5}, \frac{2}{5}]$, $\hat{z}(s,t) = \hat{y}(2s, 5t-1)$
- On $[0, \frac{1}{2}] \times [\frac{2}{5}, \frac{3}{5}]$:
  - $\hat{z}([0, \frac{1}{2}] \times [\frac{2}{5}, \frac{3}{5}]) \subset \mathrm{Sym}\underline{L}_{v_1}$
  - for $\frac{2}{5} \leq t \leq \frac{3}{5}$, $\hat{z}(0,t) = x$
  - for $0 \leq s \leq \frac{1}{2}$, $\hat{z}(s, \frac{2}{5}) = \hat{y}(2s, 1)$
  - for $0 \leq s \leq \frac{1}{2}$, $\hat{z}(s, \frac{3}{5}) = \hat{y'}(2s, 0)$
- On $[0, \frac{1}{2}] \times [\frac{3}{5}, \frac{4}{5}]$, $\hat{z}(s,t) = \hat{y'}(2s, 5t-3)$
- On $[0, \frac{1}{2}] \times [\frac{4}{5}, 1]$:
  - $\hat{z}([0, \frac{1}{2}] \times [\frac{4}{5}, 1]) \subset \mathrm{Sym}\underline{L}_{v_1}$
  - for $\frac{4}{5} \leq t \leq 1$, $\hat{z}(0,t) = x$
  - for $0 \leq s \leq \frac{1}{2}$, $\hat{z}(s, \frac{4}{5}) = \hat{y'}(2s, 1)$
  - $\hat{z}(\frac{1}{2}, 1) = z(1)$
- On $[\frac{1}{2}, 1] \times [0,1]$, $\hat{z}$ is a homotopy of braids between $\hat{z}(\frac{1}{2}, \cdot)$ and z, contained in $\mathrm{Sym}^{k+g}(\Sigma_{g,p})$. Indeed, since b is a group homomorphism, those two paths are isotopic as braids.

Now, the capping $\hat{z}$ consists of a concatenation of $\hat{y}$, $\hat{y'}$, homotopies contained in $\mathrm{Sym}\underline{L}_{v_1}$, and a braid isotopy contained in $\mathrm{Sym}^{k+g}(\Sigma_{g,p})$. Homotopies contained in $\mathrm{Sym}^{k+g}(\Sigma_{g,p})$ do not contribute to the difference of symplectic area, and as $\mathrm{Sym}\underline{L}_{v_1}$ is away from the diagonal, and (by definition) a braid isotopy does not cross the diagonal, the only contribution to the difference of action comes from $\hat{y}$ and $\hat{y'}$. Since the intersection number and the symplectic area are additive, we get that $f_{v_1,v_2}(\varphi\psi) = f_{v_1,v_2}(\varphi) + f_{v_1,v_2}(\psi)$. Moreover, since braid isotopies do not contribute to the action difference, $f_{v_1,v_2}(\varphi)$ only depends on the braid type of φ. □

Therefore, it is enough to compute the value of $\mathfrak{f}_{v_1,v_2}$ on the generators of $\mathcal{B}_{\underline{L},k}$ to express $f_{v_1,v_2}(\varphi)$.

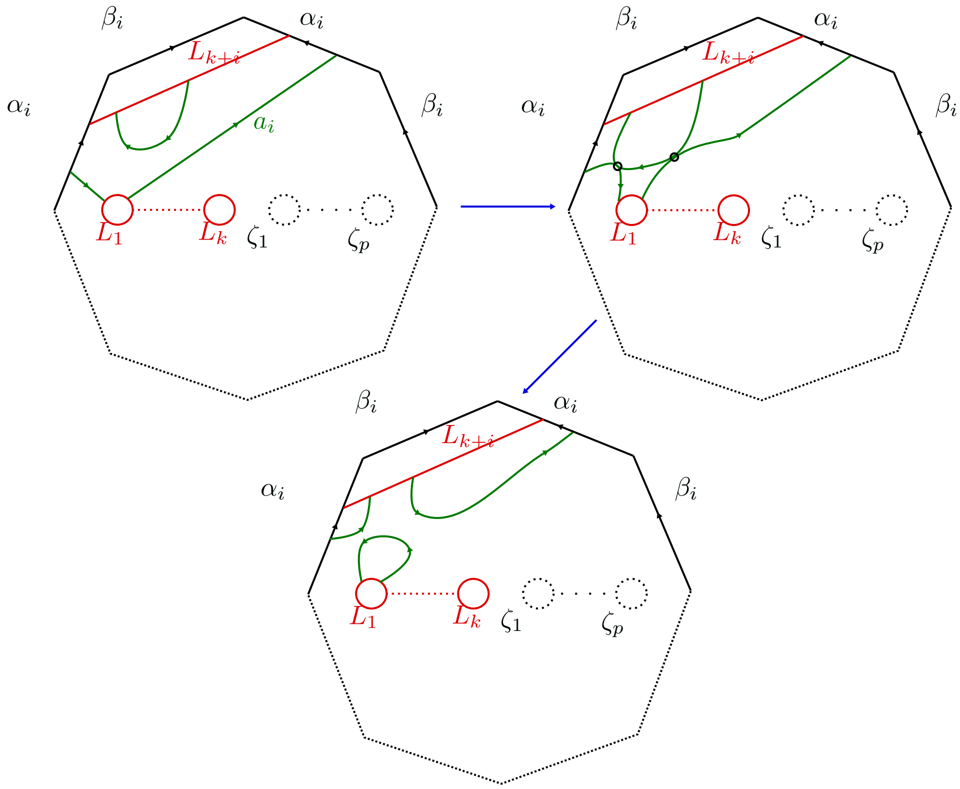


Figure 2.7: Homotopy between a_i and the constant path

Lemma 2.4.8. *The values of* $\mathfrak{f}_{v_1,v_2}$ *are the following:*

$$\mathfrak{f}_{v_1,v_2}(a_i) = 2\frac{\eta_{v_2} - \eta_{v_1}}{k+g} = -f_{v_1,v_2}(b_i^{-1}a_ib_i), \ \ \mathfrak{f}_{v_1,v_2}(\sigma_j) = \frac{\eta_{v_2} - \eta_{v_1}}{k+g},$$
$$\mathfrak{f}_{v_1,v_2}(z_j) = \frac{s_{2,j} - s_{1,j}}{k+g} = -2(\eta_{v_2} - \eta_{v_1})\frac{k+2g-1}{k+g}\frac{s_{2,j} - s_{1,j}}{s_2 - s_1}$$

Proof. To do these computations, according to Lemma 2.4.5 it is enough to produce explicit homotopies between the reference path x (which is a trivial braid) and the braid for which we know how to compute the action difference. For a_i, $b_i^{-1}a_ib_i$ and σ_j, we exhibit homotopies contained in $\Sigma_{g,p}$, so that the only contribution to the action difference comes from intersections with the diagonal. We also choose our homotopies so that the intersections with the diagonal are transverse. To be sure of their transversality, we make use the content of Section 2.3.2. We show there that the intersections we count here are transverse, and give their signs.

The result comes from counting such intersections (with sign) on Figures 2.7, 2.8 and 2.9. All the intersections appearing in those homotopies are modelled

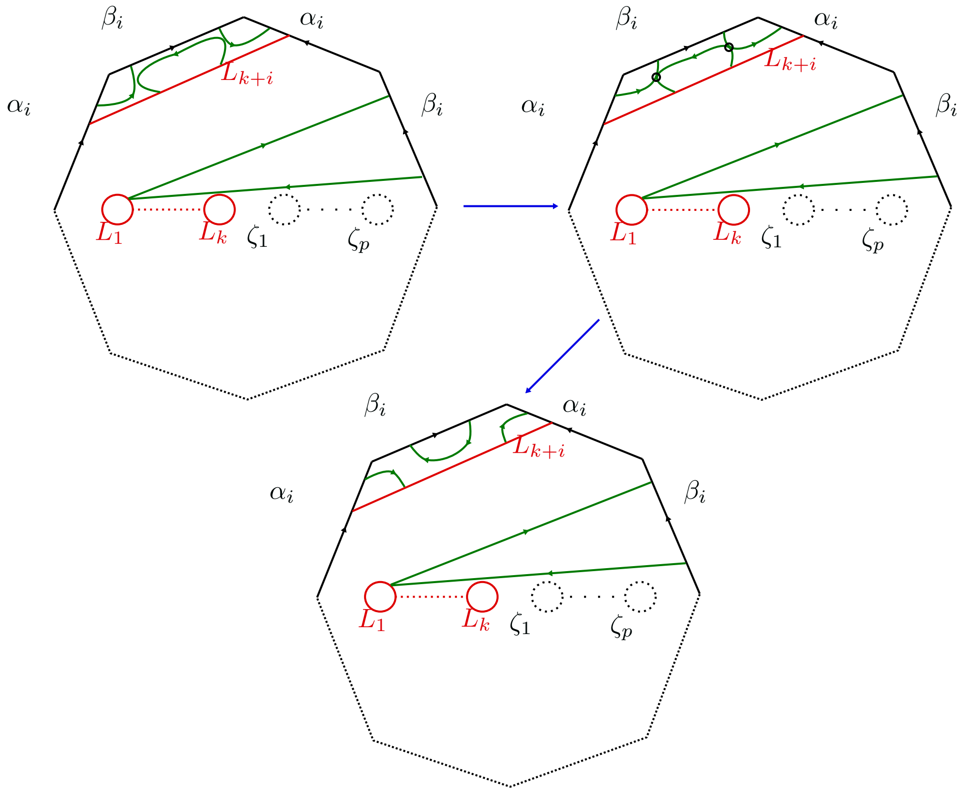


Figure 2.8: Homotopy between $b_i^{-1}a_ib_i$ and the constant path

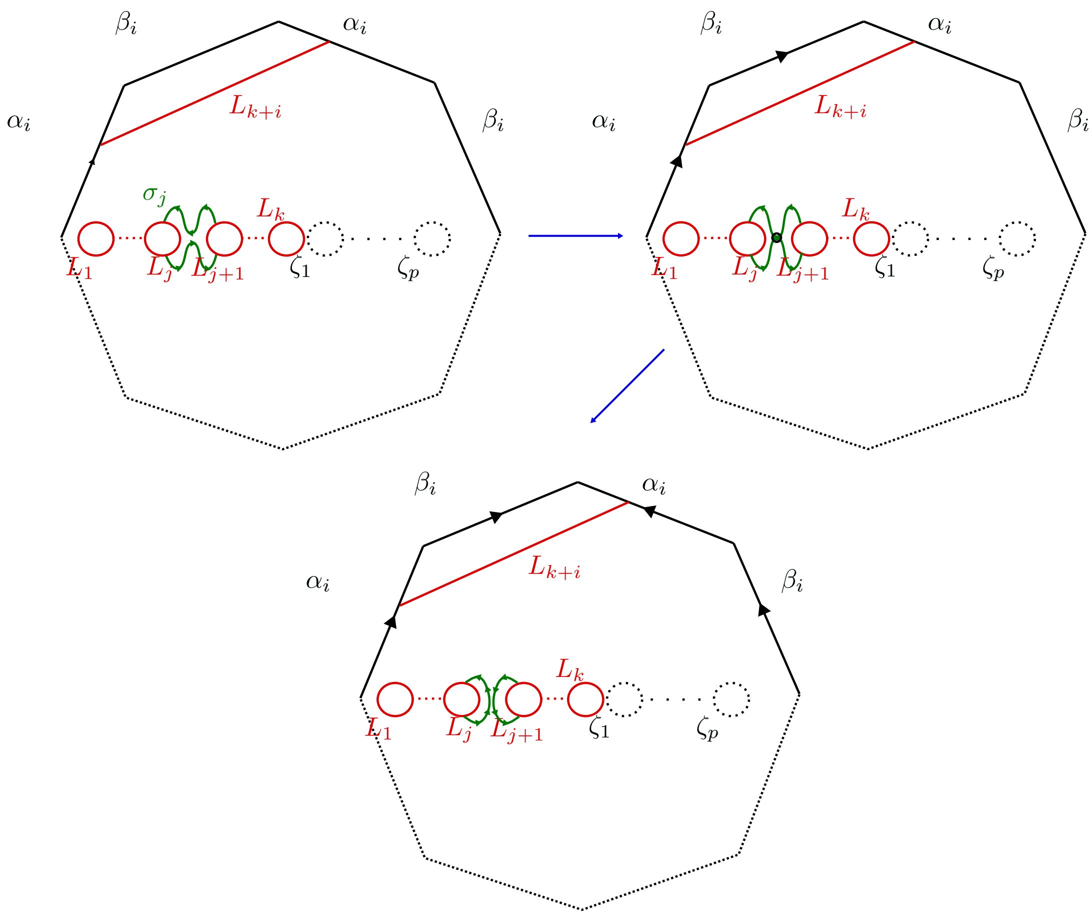


Figure 2.9: Homotopy between σ_j and the constant path

on the example of Figure 2.4, therefore we know they are transverse and can compute their signs.

For z_j, the braid becomes trivial after embedding $\Sigma_{g,p}$ into $\Sigma_g(1+s_i)$, and we can choose an isotopy to the trivial braid (without crossing) which sweeps the disc glued to the boundary component ζ_j. Therefore, the only contribution to the action difference comes from the difference of symplectic area between the two embeddings:

$$\mathfrak{f}_{v_1,v_2}(z_j) = \frac{s_{2,j} - s_{1,j}}{k+g}$$

By the monotonicity property applied to the only non contractible connected component of $\Sigma_g(1+s_i)$, we have

$$A = 1 + s_i - kA + 2\eta_{v_i}(k+2g-1)$$

and therefore $\eta_{v_2} - \eta_{v_1} = \frac{s_1 - s_2}{2(k+2g-1)}$, which implies that

$$\mathfrak{f}_{v_1,v_2}(z_j) = -2(\eta_{v_2} - \eta_{v_1})\frac{k+2g-1}{k+g}\frac{s_{2,j} - s_{1,j}}{s_2 - s_1}$$

□

Remark 2.4.9. *With this last expression for* $\mathfrak{f}_{v_1,v_2}(z_j)$*, it is easy to check that the values of* $\mathfrak{f}_{v_1,v_2}$ *are consistent with the relation of Remark 2.3.8.*

Remark 2.4.10. *The reader might be surprised by the fact that*

$$\mathfrak{f}_{(v_1,v_2)}(b_i a_i b_i^{-1}) \neq \mathfrak{f}_{(v_1,v_2)}(a_i)$$

Now, remark that the $\mathfrak{f}_{(v_1,v_2)}$ *are not defined on the* b_i*, which indeed are not elements in* $\mathcal{B}_{\underline{L}}$*.*

Remark 2.4.11. *The reader will see that the intersection numbers we compute for the generators* σ_j *here is he opposite of what we find in [52], the reason being that the* σ_j *in [52] correspond to a clockwise half-twist, while here the half-twist is counterclockwise.*

By the Hofer-Lipschitz property of link spectral invariants, we have that for all $\varphi \in \mathrm{Ham}_{\underline{L},c}(\Sigma_{g,p})$, and for any choice of $(v_1, v_2) = (s_{i,j})_{i=1,2;1\leq j\leq p} \in V \times V$:

$$\|\varphi\| \geq \frac{1}{2}|\mathfrak{f}_{v_1,v_2}(\varphi)|$$

To get the best estimate, we compute the maximum of $|\mathfrak{f}_{(s_{i,j})}(\varphi)|$ over the choice of (v_1, v_2). We have

$$(k+g)\mathfrak{f}_{v_1,v_2}(\varphi) = \frac{s_1 - s_2}{2(k+2g-1)}(2k_{gen} - k_\sigma) + \sum_{j=1}^{p-1} k_j(s_{2,j} - s_{1,j})$$

where we have decomposed $b(\varphi)$ as a product of the generators a_i, $(b_i^{-1}a_ib_i)^{-1}$, σ_j and z_j, and k_{gen} is the sum of the exponents of all the a_i and $(b_i^{-1}a_ib_i)^{-1}$ in this decomposition; k_σ is the sum of the exponents of all the σ_j, and k_j is the sum of the exponents of z_j (according to the relations between the generators, k_{gen}, k_σ and the k_j do not depend on the decomposition of $b(\varphi)$ as a product of generators). Observe that this expression is homogeneous in (v_1, v_2), i.e. $|\mathfrak{f}_{\kappa v_1,\kappa v_2}(\varphi)| = |\kappa||\mathfrak{f}_{v_1,v_2}(\varphi)|$. Therefore, the maximum has to be attained when s_1 or s_2 is maximal, i.e. equal to $(k+1)A-1$. Since permuting v_1 and v_2 does not change $|\mathfrak{f}_{v_1,v_2}(\varphi)|$, we can assume that $0 \leq s_1 \leq s_2 = (k+1)A-1$. Let $k_{max} := \max\{k_j\}$, $k_{min} := \min\{k_j\}$, and let j_{max} and j_{min} be indices such that $k_{j_{max}} = k_{max}$, and $k_{j_{min}} = k_{min}$. Then,

$$(k+g)\mathfrak{f}_{v_1,v_2}(\varphi) \leq \frac{s_1 - s_2}{2(k+2g-1)}(2k_{gen} - k_\sigma) + \sum_{s_{2,j}-s_{1,j}\geq 0} k_{max}(s_{2,j} - s_{1,j}) +$$

$$+ \sum_{s_{2,j}-s_{1,j}<0} k_{min}(s_{2,j} - s_{1,j}) \leq \frac{s_1 - s_2}{2(k+2g-1)}(2k_{gen} - k_\sigma) + s_2 k_{max} - s_1 k_{min}$$

$$= s_2\left(k_{max} - \frac{2k_{gen} - k_\sigma}{2(k+2g-1)}\right) + s_1\left(\frac{2k_{gen} - k_\sigma}{2(k+2g-1)} - k_{min}\right)$$

with equality when $s_{1,j} = \delta_{j,j_{min}} s_1$ and $s_{2,j} = \delta_{j,j_{max}} s_2$.

Similarly,

$$(k+g)\mathfrak{f}_{v_1,v_2}(\varphi) \geq \frac{s_1 - s_2}{2(k+2g-1)}(2k_{gen} - k_\sigma) + \sum_{s_{2,j}-s_{1,j}\geq 0} k_{min}(s_{2,j} - s_{1,j}) +$$

$$+ \sum_{s_{2,j}-s_{1,j}<0} k_{max}(s_{2,j} - s_{1,j}) \geq \frac{s_1 - s_2}{2(k+2g-1)}(2k_{gen} - k_\sigma) + s_2 k_{min} - s_1 k_{max}$$

$$= s_2\left(k_{min} - \frac{2k_{gen} - k_\sigma}{2(k+2g-1)}\right) + s_1\left(\frac{2k_{gen} - k_\sigma}{2(k+2g-1)} - k_{max}\right)$$

with equality when $s_{1,j} = \delta_{j,j_{max}} s_1$ and $s_{2,j} = \delta_{j,j_{min}} s_2$. Since those expressions are linear in s_1, the extremal values are attained for $s_1 = 0$ or $s_1 = s_2$. Set:

$$R = k_{max} - k_{min},\; S = k_{max} - \frac{2k_{gen} - k_\sigma}{2(k+2g-1)},\; T = \frac{2k_{gen} - k_\sigma}{2(k+2g-1)} - k_{min}$$

In the end, we get:

$$\max_{(v_1,v_2)\in V^2} |\mathfrak{f}_{v_1,v_2}(\varphi)| = \frac{(k+1)A-1}{k+g} \max\{R, S, T\} \tag{2.17}$$

and therefore

$$\|\varphi\| \geq \frac{(k+1)A-1}{2(k+g)} \max\{R, S, T\}$$

This ends the proof of Theorem 2.2.3.

The proof of Theorem 2.2.6 is very simple now. Remark first that the function $\mathfrak{f}_{v_1,v_2}$ if $\Sigma_{g,p} = \mathbb{D}$ in fact depends on only one pair of parameters, s_1 and

s_2, so we are going to adjust the notation accordingly. From Lemma 2.4.8 we find that

$$\mathfrak{f}_{s_1,s_2}(\sigma_j) = -\frac{\eta_{s_2} - \eta_{s_1}}{k}$$

and taking optimal s_1 and s_2 yields

$$\mathfrak{f}_{0,(k+1)\lambda-1}(\sigma_j) = \frac{1}{2k}\frac{(k+1)\lambda - 1}{k-1}$$

as anticipated in Section 2.2. Since f_{s_1,s_2} is a homomorphism, we may write, for any $\varphi \in \mathrm{Ham}_{\underline{L}}(\mathbb{D})$, that

$$f_{0,(k+1)\lambda-1}(\varphi) = \frac{1}{k}\frac{(k+1)\lambda - 1}{k-1}\mathrm{lk}(b(\varphi))$$

since by definition $\mathrm{lk}(\sigma_j) = 1$ for all j. We now homogenise:

$$Q_k(\varphi) := \lim_{n\to\infty}\frac{f_{0,(k+1)\lambda-1}(\varphi^n)}{n}$$

The right hand side is now a homogeneous quasimorphism on $\mathrm{Ham}_c(\mathbb{D})$ (difference of two homogeneous quasimorphisms), and since $f_{0,(k+1)\lambda-1}$ is in fact a homomorphism of groups when restricted to $\mathrm{Ham}_{\underline{L}}(\mathbb{D})$, we finish the proof of Theorem 2.2.6:

$$\forall\varphi \in \mathrm{Ham}_{\underline{L}}(\mathbb{D}),\ Q_k(\varphi) := \lim_{n\to\infty}\frac{f_{0,(k+1)\lambda-1}(\varphi^n)}{n} = \\ = f_{0,(k+1)\lambda-1}(\varphi) = \frac{1}{2k}\frac{(k+1)\lambda - 1}{k-1}\mathrm{lk}(b(\varphi))$$

2.5 Hofer norms for braid groups on surfaces with boundary

Let us start this Section proving that the quantities $\|\cdot\|_{\underline{L}}$ are at least pseudonorms on the images of b.

Lemma 2.5.1. *For all $g_1, g_2 \in \mathcal{B}_{\underline{L}}$, $\|g_1\| = \|g_1^{-1}\|$ and $\|g_1g_2\| \leq \|g_1\| + \|g_2\|$.*

Proof. The first point is obvious since for all $\varphi \in \mathrm{Ham}_c(\Sigma_{g,p},\omega)$, $\|\varphi\| = \|\varphi^{-1}\|$ and $b(\varphi) = b(\varphi^{-1})^{-1}$. For the second one,

$$\|g_1g_2\|_{\underline{L}} = \inf_{\varphi\in\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p},\omega), b(\varphi)=g_1g_2}\|\varphi\| \leq \inf_{\varphi_i,\in\mathrm{Ham}_{\underline{L}}(\Sigma_{g,p},\omega), b(\varphi_i)=g_i}\|\varphi_1\varphi_2\|$$

and applying triangular inequality

$$\|g_1g_2\|_{\underline{L}} \leq \|\varphi_1\| + \|\varphi_2\|,\ \forall\,\varphi_i \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p},\omega) \text{ with } b(\varphi_i) = g_i$$

so that taking the infimum first on φ_1 and then on φ_2 we conclude. □

Now, Chen in [15] proved non degeneracy of the pseudonorms we defined in [51]. We aim to show how Chen's proof may be adapted to extend his theorem to:

Theorem 2.5.2. *Let $\underline{L}$ be a pre-monotone link on $\Sigma_{g,p}$, a compact symplectic surface of genus g and with p boundary components. Then there exists an $\varepsilon > 0$, only depending on $\underline{L}$, such that if $\varphi, \psi \in Ham_{\underline{L}}(\Sigma_{g,p})$ are such that $d_H(\varphi, \psi) < \varepsilon$, then $b(\varphi) = b(\psi)$.*

This theorem in turn implies that the pseudonorms we defined are non degenerate.

Corollary 2.5.3. *If $(\varphi_i) \subset Ham_{\underline{L}}(\Sigma_{g,p})$ is a sequence of Hamiltonian diffeomorphisms preserving $\underline{L}$, and $\|\varphi_i\| \to 0$, then there exists a positive integer n such that for every $i > n$ the braid $b(\varphi_i)$ is trivial.*

We are now going to adapt Chen's proof to our setting.

In [15], the author considers Hofer-close Hamiltonian diffeomorphisms φ and ψ. Choosing generating Hamiltonians for both of them and perturbing them gives rise to the associated Floer complexes, and continuation maps between them. If the generating Hamiltonians are $\mathcal{C}^0$-close, one may furthermore prove that the pseudoholomorphic curves appearing in the continuation maps do not intersect the diagonal of the symmetric product, nor the divisor

$$D = z + \mathrm{Sym}^{k-1}(\mathbb{S}^2)$$

where z is the north pole of the sphere (outside the image of the embedding $\mathbb{D} \hookrightarrow \mathbb{S}^2$. The fact that the two Hamiltonians may be chosen to be close of course follows from the definition of the Hofer norm. One may then use these pseudoholmorphic curves to produce a braid isotopy between chosen representatives of $b(\varphi)$ and $b(\psi)$ using the pseudoholomorphic curves of the continuation maps.

In our case we wish to prevent pseudoholomorphic curves from intersecting the diagonal and exiting the surface $\Sigma_{g,p}$: we glue discs of appropriate areas to the boundary components, so that $\underline{L}$ becomes monotone, and consider the centres ζ_i, $i = 1, \dots, p$ of the discs we just glued. Define the divisors

$$D_i = \zeta_i + \mathrm{Sym}^{k+g-1}(\Sigma_g) \tag{2.18}$$

if $u : \mathbb{D} \to \mathrm{Sym}^k(\Sigma_g)$ is a pseudoholomorphic disc with Lagrangian boundary conditions on the link $\underline{L}$, if $[u] \cdot D_i = 0$ for each i then the image does not leave the image of the embedding $\Sigma_{g,p} \hookrightarrow \Sigma_g$, by positivity of the intersections between (pseudo)holomorphic submanifolds: the Hamiltonian vector field is in fact identically zero close to the ζ_i.

As in [15], we then decompose the differential in the Floer complex as a sum of contributions, each of them counting pseudoholomorphic discs with fixed intersection number with the diagonal Δ and the divisors D_i. The function ∂_{00}, counting pseudoholomorphic discs not intersecting Δ or any of the divisors D_i is a differential, and we consider the complex $(CF(\underline{L}, H), \partial_{00})$. As Chen points out, it is possible to do the same with continuation maps: if K is a compactly supported Hamiltonian generating $\psi \in \mathrm{Ham}_{\underline{L}}(\Sigma_{g,p})$, let $h : CF(\underline{L}, H) \to CF(\underline{L}, K)$ be the continuation map associated to a regular homotopy between H and K. One may consider h_{00}, obtained from h counting contribution of pseudoholomorphic curves not intersecting Δ or any D_i: it turns out to be a chain homotopy between the two complexes $(CF(\underline{L}, H), \partial_{00})$ and $(CF(\underline{L}, K), \partial_{00})$. The arguments used in [15] to prove existence of pseudoholomorphic curves contributing to h_{00} (assuming that $\|H - K\|_{(1,\infty)}$ is small) carry over to our case: they in fact either involve local considerations around the Lagrangians, or use monotonicity of the link which we have now proved (see Proposition 2.3.1). As a consequence, one may follow through the proof of Theorem 1 in [15], which provides the braid isotopy we are looking for: such an isotopy is given by gluing a pseudoholomorphic curve contributing to h_{00} with a disc whose image does not leave the torus $\mathrm{Sym}^{k+g}\underline{L}$. This completes the proof of Theorem 2.5.2.

Chapter 3

A Filtration in Linking Numbers

3.1 Introduction

In this chapter we describe one of the main results of this thesis, namely the existence of a filtration in the Morse theory for a compactly supported Hamiltonian diffeomorphism that keep tracks of the linking number of pairs of orbits. We shall start describing the works of Patrice Le Calvez, and then use his setup to prove this Theorem. The main tools used here are the classical theory of twist maps and generating functions; standard references for the field are the works of Le Calvez themselves [45] and [44], the works of Viterbo (see for instance [73]) and those of Théret [69]. At the end of the chapter we are also going to discuss how the Morse picture relates to the Floer one: a good technical prerequisite here is the beginning of the series of papers by Hofer, Wysocki and Zehnder [32], [31]; one can find a good and succint overview in the work of Connery-Grigg [19].

The main Theorem may be stated as follows:

Theorem 3.1.1. *Let φ be a compactly supported Hamiltonian diffeomorphism of the plane with its standard symplectic form $dx \wedge dy$. Assume φ is non degenerate on the interior of its support. Then for any generating function quadratic at infinity $S : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ for φ, there exists a non degenerate quadratic form Q on $\mathbb{R}^l$, a Riemannian metric g on $\mathbb{R}^{2+k+l}$ such that the pair $(S \oplus Q, g)$ is both Palais-Smale and Morse-Smale, and making (an extension of) the function*

$$I : CM(S \oplus Q, g; \mathbb{Z}) \otimes CM(S \oplus Q, g; \mathbb{Z}) \to \mathbb{Z},$$

$$I(x \otimes y) := \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & x \neq y \\ -\left\lceil \frac{CZ(\gamma_x)}{2} \right\rceil & x = y \end{cases}$$

into an increasing filtration of the tensor complex. Here, x and y are critical points of $S \oplus Q$, and γ_x, γ_y are the associated fixed points of φ.

Moreover, an analogous filtration exists for the Floer complex of φ, and it behaves well with respect to the pair of pants product.

3.2 Setup: Le Calvez-type generating functions

All the material in this section comes from the works of Le Calvez's [45], [44].

Let $\varphi \in \mathrm{Ham}_c(\mathbb{D})$. As Patrice Le Calvez points out in [45], it is possible to describe φ as a composition of Hamiltonian twist maps of the plane, which therefore have generating functions in the sense of Section 1.3.1. A way of doing so is for instance choosing a Hamiltonian isotopy $(\varphi_t)_{t\in[0,1]}$ ending at φ, and cutting it into pieces $\varphi_0, \dots, \varphi_{n-1}$ so that each φ_i is $\mathscr{C}^1$-close to the identity anf

$$\varphi = \varphi_{n-1} \circ \cdots \circ \varphi_0$$

If the φ_i are close enough to the identity and

$$R : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (y, -x) \tag{3.1}$$

is the positive rotation, then $\varphi_i \circ R^{-1}$ is a twist map (Definition 1.3.5) for all i. We thus obtain a decomposition of φ as product of twist maps

$$\varphi = (\varphi_{n-1} \circ R^{-1}) \circ R \circ (\varphi_{n-2} \circ R^{-1}) \circ R \circ \cdots \circ (\varphi_0 \circ R^{-1}) \circ R \tag{3.2}$$

Remark 3.2.1. *Le Calvez in [44] uses Dehn twist instead of rotations. This is not going to affect the properties we exploit here in any way. What is important is the twist condition, that here still holds.*

Write $\Phi_{2i} := R$ and $\Phi_{2i+1} := \varphi_i \circ R^{-1}$ in order to simplify the notation:

$$\varphi = \Phi_{2n-1} \circ \cdots \circ \Phi_0 \tag{3.3}$$

Now, we let $h_i : \mathbb{R}^2 \to \mathbb{R}$ be the generating function for Φ_i. Recall that by definition then the following equations are verified:

$$\forall (x, y), (x', y') \in \mathbb{R}^2,\ \Phi_i(x, y) = (x', y') \Leftrightarrow \begin{cases} y = -\partial_1 h(x, x') \\ y' = \partial_2 h(x, x') \end{cases} \tag{3.4}$$

The critical points of h_i, as expected, are in a bijection with the fixed points of Φ_i. Following Le Calvez's notation, we are going to write

$$\begin{cases} g(x, x') = -\partial_x h(x, x') \\ g'(x, x') = \partial_{x'} h(x, x') \end{cases}$$

One can then take the sum of the h_i to find a function whose critical points (equivalently, fixed points of its gradient flow) are in bijection with the fixed points of φ. We define a function

$$h : E := \mathbb{R}^{2n} \to \mathbb{R},\ h(x) = \sum_{i=0}^{2n-1} h_i(x_i, x_{i+1}) \tag{3.5}$$

where we set $x_{2n} := x_0$. We endow $\mathbb{R}^{2n}$ with the standard Euclidean scalar product, let us call it g; let ξ be the negative gradient vector field defined by the pair (h, g)

$$\xi = -\nabla_g h \tag{3.6}$$

The tool we use to calculate the linking number between fixed points is a function defined on an open subset of $\mathbb{R}^{2n}$. If

$$V = \left\{ x \in \mathbb{R}^{2n} | x_i \neq 0 \ \forall i \in \mathbb{Z} \right\} \tag{3.7}$$

we define

$$L(x) = \frac{1}{4} \sum_{i=0}^{2n-1} (-1)^i \mathrm{sgn}(x_i)\mathrm{sgn}(x_{i+1}) \in \mathbb{Z} \tag{3.8}$$

This function admits a continuous extension to the subset

$$W = \left\{ x \in \mathbb{R}^{2n} | \forall i \in \mathbb{Z},\ x_i = 0 \Rightarrow x_{i-1}x_{i+1} > 0 \right\} \tag{3.9}$$

A first result is the following Lemma:

Lemma 3.2.2. *Let x^0 and x^1 be critical points of h (equivalently, $\xi(x^i) = 0$). Then,*

$$L(x^0 - x^1) = \frac{1}{2} \mathrm{lk}(\gamma_{x^1}, \gamma_{x^0}) \tag{3.10}$$

On the right hand-side, $\gamma_{x^i} \in \mathrm{Fix}(\varphi)$ is the fixed point (or indifferently, 1-periodic orbit) represented by the critical points x^i of h.

Remark 3.2.3. *The factor of one half is absent in [44]. We introduce it here because we choose a different normalisation for the linking number. The reason for this difference is simply explained: in Chapter 2 we work with braids which are not necessarily pure, and within that setup the algebraic definition (mapping the generators of the braid group to 1) seems more convenient. Furthermore, it makes the linking number agree with the intersection numbers of cappings with the fat diagonal in the symmetric product. This normalisation differs by a factor of 2 from the one of Le Calvez: to see it for instance remark that the linking number of the braid*

$$t \mapsto \left[0, \frac{1}{2} \exp(2\pi i t) \right] \in Sym^2\mathbb{D}$$

is 1 according to Le Calvez's conventions, but with our normalisation it is 2 since it corresponds to the square of the generator of $\mathcal{B}_2$.

Remark 3.2.4. *A consequence of this lemma and the fact that L takes values in the set $\left\{ -\left\lfloor \frac{n}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$ is that the length of the decomposition of ϕ necessarily depends on the maximal (in absolute value) linking number of two different fixed points.*

The first main result is the following theorem:

Theorem 3.2.5 (Le Calvez, [45]). *Let $x^i \in E$, $i = 0, 1$, and denote by $x^i(t)$ the images of x^i under the flow of ξ. Then the function*

$$t \mapsto L(x^0(t) - x^1(t)) \tag{3.11}$$

is defined and continuous outside a finite set of points, and if it is not defined for a $t \in \mathbb{R}$ then

$$\lim_{\varepsilon \to 0^+} L(x^0(t-\varepsilon) - x^1(t-\varepsilon)) < \lim_{\varepsilon \to 0^+} L(x^0(t+\varepsilon) - x^1(t+\varepsilon)) \tag{3.12}$$

This theorem is the first step to define the filtration. The goal is, for any S generating function for φ and g is a Riemannian metric on the associated vector bundle, to define a function

$$I : CM(S, g; \mathbb{Z}) \otimes CM(S, g; \mathbb{Z}) \to \mathbb{Z} \tag{3.13}$$

which is increasing along the differential. The Theorem tells us that, if we take h as generating function (it is yet to be seen in which sense it is one) whenever $y \notin \partial x$, we have the inequality

$$I(x \otimes y) \leq I(\partial x \otimes y) \tag{3.14}$$

What remains to define a filtration is removing the assumption that $y \notin \partial x$, or equivalently defining a notion of self-linking number for a fixed point of φ which is consistent with the linking numbers of all other pairs of fixed points in the Morse complex. To carry out this operation, we need to understand the behaviour of L under the linearised dynamics of ξ. Before carrying this out, we would like to explain how this function h is in fact a generating function in the classical (à la Viterbo) sense, to fix the ideas. This fact is relevant as it will be exploited in constructing the filtration for any (reasonable) generating function in the following.

Le Calvez's h is a generating function

We show that this h is a generating function for φ in the sense of definitions 1.3.10 and 1.3.11 as seen through the equations (1.12). This is the content of Corollary 3.2.7, stated below.

A point $(x, y) \in \mathbb{R}^2$ is mapped to (X, Y) if and only if there are (unique) points $(x_i, y_i) \in \mathbb{R}^2$, $i \in \{0, \ldots, 2n\}$ such that $(x_0, y_0) = (x, y)$, $(x_{2n}, y_{2n}) = (x', y')$, $\varphi_i(x_{i_1}, y_{i-1}) = (x_i, y_i)$. This last condition is verified if and only if, since h_i generates f_i,

$$\begin{cases} y_{i-1} = -\partial_{x_{i-1}} h_i(x_{i-1}, x_i) = g_i(x_{i-1}, x_i) \\ y_i = \partial_{x_i} h_i(x_{i-1}, x_i) = g_i'(x_{i-1}, x_i) \end{cases}$$

Let us assume that the decomposition ends in $R^{-1} \circ R$ (we are not losing in generality, we are essentially adding a trivial factor in the decomposition of φ

into $\mathscr{C}^1$-small factors); the order is important , since we want our maps in the decomposition to twist positively and negatively alternatively. The decomposition of φ more concretely will have to look like this:

$$R^{-1} \circ R \circ \varphi_{n-1} \circ R^{-1} \circ \cdots \varphi_0 \circ R^{-1} \circ R$$

where each φ_i is $\mathscr{C}^1$-small.

Remember now that R is generated by $(x, x') \mapsto -xx'$ and R^{-1} by $(x, x') \mapsto xx'$. A Le Calvez generating function associated to this decomposition is therefore:

$$h : \mathbb{R}^{2n+2} \to \mathbb{R},\ (x_0, \dots, x_{2n+1}) \mapsto -x_0 x_1 + \sum_{i=1}^{2n-1} h_i(x_i, x_{i+1}) - x_{2n} x_{2n+1} + x_{2n+1} x_0$$

We isolated the first and the last two terms because they are the ones we are going to differentiate in the following.

We now define the projection:

$$\mathbb{R}^{2n+2} \to \mathbb{R}^2,\ (x_0, \dots, x_{2n+1}) \mapsto (x_0, x_{2n+1})$$

Requiring the vertical differential to be 0 is then equivalent to asking for $\xi_i = 0$ for $i = 1, \dots, 2n$, i.e. (compare [44]) for

$$\Phi_{i-1}(x_{i-1}, y_{i-1}) = (x_i, y_i),\ \text{for } i = 1, \dots, 2n$$

A simple computation then gives:

$$\begin{cases} x_{2n+1} - x_1 = \partial_{x_0} h(x_0, \dots, x_{2n+1}) \\ -x_{2n} + x_0 = \partial_{x_{2n+1}} h(x_0, \dots, x_{2n+1}) \end{cases} \tag{3.15}$$

Let us show how this is enough to conclude. The vanishing of the vertical differential ensures that $x_{2n+1} = y_{2n}$: in fact

$$\partial_{x_{2n}} h = 0 \Leftrightarrow x_{2n+1} = g'_{2n-1}(x_{2n-1}, x_{2n})$$

but, on the other side, we know ($\xi_{2n} = 0$) that

$$\Phi_{2n-1}(x_{2n-1}, y_{2n-1}) = (x_{2n}, y_{2n})$$

that which is verified if and only if

$$\begin{cases} y_{2n-1} = g_{2n-1}(x_{2n_1}, x_{2n}) \\ y_{2n} = g'_{2n-1}(x_{2n_1}, x_{2n}) \end{cases}$$

and the second equation allows us to conclude that

$$x_{2n+1} = y_{2n}$$

as desired.

Going back to the formalism of the introduction,write

$$(X,Y) := \varphi(x,y)$$

then $(x_{2n}, y_{2n}) = (X, Y)$. The equations (3.15) above then become

$$\varphi(x_0,y_0) = (X,Y) \Leftrightarrow \begin{cases} \partial^V h = 0 \\ X - x = \partial_Y h(x_0 = x, x_1, \dots, x_{2n}, x_{2n+1} = Y) \\ Y - y = -\partial_x h(x_0 = x, x_1, \dots, x_{2n}, x_{2n+1} = Y) \end{cases}$$

and this is precisely the same equivalence as in (1.12). This shows that h is indeed a generating function for φ in the classical sense, but we do not know yet if we can do Morse theory with it: we need the Palais-Smale condition. A first step is proving that h can be made quadratic at infinity by a linear map of $\mathbb{R}^{2n}$ that is also a gauge equivalence (Definition 1.3.17).

Quadraticity at infinity

If φ has compact support, it is easy to see that every term in the decomposition can be assumed to coincide with the positive or the negative rotation outside a compact set (see [45]). Write $\varphi = \Phi_{2n+1} \circ \cdots \circ \Phi_0$ as above. The generating function associated to the decomposition, outside of a compact set of $\mathbb{R}^{2n+2}$, coincides with

$$h_\infty(x_0, \dots, x_{2n+1}) = \sum_{i=0}^{2n+1} (-1)^{i+1} x_i x_{i+1}$$

which is a genuine quadratic form whose kernel has dimension 2: the Gram matrix associated to h is a so-called Jacobi matrix

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 1 & & & 0 \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ 0 & & & -1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Such a bilinear form clearly has a two dimensional kernel, spanned by ((e_j) is here the canonical basis of $\mathbb{R}^{2n+2}$)

$$\sum_{i=0}^{n} e_{2i}, \sum_{i=0}^{n} e_{2i+1}$$

Completing the squares as follows gives a linear diffeomorphism of $\mathbb{R}^{2n+2}$ that preserves the fibres of the projection

$$\mathbb{R}^{2n+2} \to \mathbb{R}^2, (x_0; \dots, x_{2n+1}) \mapsto (x_0, x_{2n+1})$$

and that makes h non degenerate on such fibres.

Lemma 3.2.6. *Up to a linear gauge equivalence,*

$$h_\infty(x_0, \dots, x_{2n+1}) = \frac{1}{4}\sum_{i=1}^{2n}(-1)^i x_i^2$$

Proof. Let us remark that

$$-x_{2i}x_{2i+1} + x_{2i+1}x_{2i+2} + x_{2n+1}x_{2i} = (x_{2i+2} - x_{2i})(x_{2i+1} - x_{2n+1}) + x_{2i+2}x_{2n+1}$$

We apply this identity inductively to show

$$h_\infty(x_0, \dots, x_{2n+1}) = \sum_{i=0}^{n-1}(x_{2i+2} - x_{2i})(x_{2i+1} - x_{2n+1})$$

Every summand in the right hand side can be easily checked to be equal to

$$\frac{1}{4}\left\{(x_{2i+2} - x_{2i} + x_{2i+1} - x_{2n+1})^2 - (x_{2i+2} - x_{2i} - x_{2i+1} + x_{2n+1})^2\right\}$$

so that

$$h_\infty(x_0, \dots, x_{2n+1}) = \frac{1}{4}\sum_{i=0}^{n-1}\left\{(x_{2i+2} - x_{2i} + x_{2i+1} - x_{2n+1})^2 - (x_{2i+2} - x_{2i} - x_{2i+1} + x_{2n+1})^2\right\}$$

We define the endomorphism by

$$\begin{aligned} x_{2i+1} &\mapsto x_{2i+2} - x_{2i} + x_{2i+1} - x_{2n+1} \\ x_{2i+2} &\mapsto x_{2i+2} - x_{2i} - x_{2i+1} + x_{2n+1} \end{aligned}$$

for i between 0 and $n-1$. We map x_0 and x_{2n+1} to themselves as promised. The composition of h with this linear, fibre-preserving diffeomorphism has the shape we were looking for. □

Corollary 3.2.7. *Up to a fibre preserving diffeomorphism generating functions of Le Calvez type are quadratic at infinity, of signature n for a decomposition of length $2n+2$.*

Palais-Smale condition Let $\psi \in \mathrm{GL}(\mathbb{R}^{2n})$ be the linear map defined in the previous section, and let g be the standard Euclidean product on $\mathbb{R}^{2n}$. Le Calvez's results (Theorem 3.2.5, the existence of the Dominated Splitting and the Homothety Law (3.21) discussed in the next section) hold for the pair (h, g), hence also for the pair $(h \circ \psi, \psi^* g)$, since there is a clear bijection between the flow lines of the two pairs. We are now interested in showing that $(h \circ \psi, \psi^* g)$ is Palais-Smale, to achieve the good definition of the Morse complex (up to small perturbation of the metric) on the one hand, and to preserve the Lyapunov property of the linking number on the other.

The Palais-Smale property for $(h \circ \psi, g)$ is clear because $h \circ \psi$ is quadratic at infinity. This implies what we want, by the following obvious lemma:

Lemma 3.2.8. *Let f be a function on $\mathbb{R}^k$, $A \in \mathrm{GL}_k(\mathbb{R})$, and g an inner product on $\mathbb{R}^k$. If (f, g) satisfies the Palais-Smale condition, so does (f, A^*g).*

Proof. The conclusion is very easy to see: given a sequence of points x_j such that $|f(x_j)|$ is bounded and $\|\nabla^{A^*g} f(x_j)\|_{A^*g} \to 0$, we have to check that

$$\|\nabla^g f(x_j)\|_g \to 0 \tag{3.16}$$

and then apply the fact that (f, g) is Palais-Smale to deduce the existence of a convergent subsequence. The limit (3.16) is readily verified by standard estimates obtained using the operator norm of A. □

Remark 3.2.9. *This proof clearly works in the more general setting of diffeomorphisms whose differential is bounded.*

In Section 3.3.1 we are going to apply the construction from Section 1.3.2 to therefore define the Morse complex $CM(h, g)$, for some Riemannian metric g on $\mathbb{R}^{2n}$.

3.3 Definition of the filtration

Le Calvez's work [44] provides us with another useful tool, a "dominated splitting" (the original wording being "décomposition subordonnée") of TE. The content of the following paragraph may be found in [44, Proposition 3.2.1].

Let $\psi : \mathbb{R} \times E \to E$ be the flow of ξ (the flow being complete is a result of Le Calvez). Define, for $j \in \left\{-\left\lfloor \frac{n}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, the subset of $T_xE = E$

$$E_j(x) = \left\{ v \in T_xE \,|\, \forall t \in \mathbb{R},\ L(d_x\psi^t(v)) = j \right\} \cup \{0\} \tag{3.17}$$

The first result one needs to be aware of is that $E_j(x)$ is in fact a vector subspace of E. This is not immediately clear, since one from the definition only has invariance under scalar multiplication. Le Calvez also computes its dimension: if n is even and $j = \pm\left\lfloor \frac{n}{2} \right\rfloor$ then $\dim E_j(x) = 1$, in all other cases $\dim E_j(x) = 2$. The E_j form a decomposition of the tangent bundle in the sense that

$$T_xE = \bigoplus_{j \in \left\{-\left\lfloor \frac{n}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}} E_j(x) \tag{3.18}$$

for all $x \in E$.

The decomposition is subordinated to the flow in the sense that it is compatible with it: for all $x \in E$ and $t \in \mathbb{R}$ we have the equality

$$d_x\psi^t E_j(x) = E_j(\psi^t(x)) \tag{3.19}$$

We also define, for each j in the image of L, the spaces

$$E_j^+(x) = \bigoplus_{j \le k \le \left\lfloor \frac{n}{2} \right\rfloor} E_k(x), \quad E_j^-(x) = \bigoplus_{j \ge k \ge -\left\lfloor \frac{n}{2} \right\rfloor} E_k(x) \tag{3.20}$$

Equivalently, $E_j^+(x)$ (resp. $E_j^-(x)$) is the set of vectors u in T_xE such that

$$L(d_x\psi^t.u) \geq j \text{ (resp. } L(d_x\psi^t.u) \leq j) \ \forall t \in \mathbb{R}$$

The last property will prove crucial in the next section, as we are going to use it to prove the existence of said filtration.

Proposition 3.3.1. *Let $u_j \in E_j(x)$ and $u_k \in E_k(x)$ be two vectors of norm 1. We assume that $j < k$. Then, along the linearised flow of ξ, the norm of u_j shrinks much faster than the one of u_k does. More formally, there exists a $\lambda \in (0,1)$ and a positive constant C such that for all positive time $t > 0$ we have:*

$$\frac{\|d_x\psi^t u_j\|}{\|d_x\psi^t u_k\|} \leq C\lambda^t \tag{3.21}$$

Here the symbol $\|\cdot\|$ indicates the norm of the standard Euclidean product on E, which we denoted g above. We call (3.21) the "homothety law".

This is proved in [44, Lemma 3.2.2].

Remark 3.3.2. *When comparing with [44], the reader should be mindful of the fact that here ξ is the negative gradient of h, in contrast with Le Calvez's convention. In his conventions, the function L is decreasing along pairs of flow lines, and in Equation 3.21 one needs to swap numerator and denominator.*

3.3.1 Existence of the filtration

The main idea we are going to exploit here is that the way the linearised flow of ξ at a critical point changes the norms of unit vectors is governed by two phenomena. The first, which is classical, is simply given by the eigenvalues of the Hessian of h at the critical point. The second phenomenon is the homothety law contained in Equation 3.21. The two phenomena will turn out to be clearly not independent, and their interplay will yield the value that I should have on the diagonal; equivalently, it will yield a well-defined notion of self-linking number of a fixed point of φ. We assume here for simplicity that φ is non degenerate (and h Morse as a consequence): the Morse-Bott case will be treated in the Appendix A.

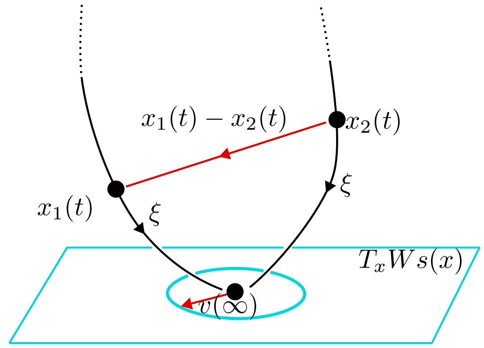
We start by fixing a critical point x of h, and we take two other points $x^1, x^2 \in E$ such that

$$\lim_{t\to+\infty} x^i(t) = x, \ x = 1, 2. \tag{3.22}$$

and which belong to different gradient lines, i.e.

$$\forall t \in \mathbb{R}, x^1(t) \neq x^2$$

Remark 3.3.3. *This allows for x^1 or x^2 to be a critical point, but not both.*



We take the difference vector and we normalise it:

$$\mathbb{R} \to \mathbb{S}^{2n-1}, t \mapsto v(t) := \frac{x^1(t) - x^2(t)}{\|x^1(t) - x^2(t)\|} \tag{3.23}$$

We denote by $v(\infty)$ the limit-set at positive infinity:

$$v(\infty) = \left\{ z \in \mathbb{S}^{2n-1} | \exists (t_k) \subset \mathbb{R}, t_k \to +\infty, v(t_k) \to z \right\} \subseteq T_xE \tag{3.24}$$

The first lemma shows that in fact any vector in $v(\infty)$ is tangent to the stable manifold of ξ at x.

Lemma 3.3.4. $v(\infty) \subset T_xW^s(x)$

Proof. For this proof, we assume without loss of generality that v admits a limit at $+\infty$ (i.e. $v(\infty)$ is a point). The fact that this is the case is going to be proven in the following lemma.

It suffices to check that the quantity

$$\frac{1}{\|x^1(t) - x^2(t)\|^2} \mathcal{H}h(x)(x^1(t) - x^2(t), x^1(t) - x^2(t)) \tag{3.25}$$

where $\mathcal{H}h(x)$ is the Hessian of h at x, is positive at the limit. Let us consider a Morse chart centred at x, and assume that the metric in the chart is Euclidean: we may assume this if we allow for changes in the Riemannian metric on E, which we now do.

Given that $x^i(t) \to x$ as $t \to \infty$, we may assume without loss of generality that $x^i(t)$ (for $i = 1, 2$) and $x^1(t) - x^2(t)$ belong in the Morse chart. Working in said chart, we are now going to prove the estimates

$$\|x^i(t) - x\| \sim C^i(x,t)e^{-\mu_{x^i}t}, \ i = 1, 2$$

where the μ_{x^i} are the lowest eigenvalues of $\mathcal{H}h(x)$ appearing in an expression of $x^i(t)$ in the Morse chart around x. To simplify the notation, let us suppose without loss of generality that $x = 0$. Denote by ϕ the Morse chart around x.

$$\|x^i(t)\|_E = \frac{\|x^i(t)\|_E}{\|\phi^{-1}x^i(t)\|} \|\phi^{-1}x^i(t)\| \sim$$
$$\sim \frac{\|x^i(t)\|_E}{\|\phi^{-1}x^i(t)\|} \|x^i\| e^{-\mu_{x^i}t}$$

The terms $C^i(x,t) := \frac{\|x^i(t)\|_E}{\|\phi^{-1}x^i(t)\|}$ are positive and bounded in t, as follows from a Taylor expansion to the first order of ϕ^{-1}. They do not tend to 0 as $t \to +\infty$ since ϕ^{-1} has invertible differential. Furthermore, as $x^i(t) \to x$, both eigenvalues μ_{x^i} are positive.

We can use this description to find a function $C^{1,2}(\cdot)$ defined for large times, such that $\|x^1(t) - x^2(t)\| \sim C^{1,2}(t)e^{-\mu t}$. To do so, simply apply cosine formula, keeping in mind that the cosine of the angle spanned by the vectors $(x^1(t) - x, x^2(t) - x)$ cannot be 1 since the flow of the gradient is conjugated by a diffeomorphism to a radial vector field. If $\delta(t)$ is the cosine, we have

$$\|x^1(t) - x^2(t)\|^2 = \|x^1(t) - x\|^2 + \|x^2(t) - x\|^2 - 2\delta(t)\|x^1(t) - x\|\|x^2(t) - x\| \sim$$
$$\sim C^1(x,t)^2 e^{-2\mu_{x^1} t} + C^2(x,t)^2 e^{-2\mu_{x^2} t} - 2\delta(t)C^1(x,t)C^2(x,t)e^{-(\mu_{x^1}+\mu_{x^2})t} =$$
$$= [C^1(x,t)^2 + C^2(x,t)^2 e^{-2(\mu_{x^2}-\mu_{x^1})t} - 2\delta(t)C^1(x,t)C^2(x,t)e^{-(\mu_{x^2}-\mu_{x^1})t}]e^{-2\mu_{x^1} t}$$

For large t

$$C^1(x,t)^2 + C^2(x,t)^2 e^{-2(\mu_{x^2}-\mu_{x^1})t} - 2\delta(t)C^1(x,t)C^2(x,t)e^{-(\mu_{x^2}-\mu_{x^1})t} > 0$$

and it does not tend[1] to 0, as $\delta < 1 - \varepsilon$ for large enough t. Without loss of generality, we assume $\mu_{x^2} - \mu_{x^1} \geq 0$: the quantity above is then bounded as a function of t. We denote it by $C^{1,2}(t)$.

Now, let $(v_1, \ldots, v_{2n})$ be a basis of E in eigenvectors for $\mathcal{H}h(x)$. Let λ_i be the eigenvalue of v_i. We can write in this basis the directions $x^1(t) - x$ and $x^2(t) - x$, defining functions $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}$.

$$\frac{x^1(t) - x}{\|x^1(t) - x\|} = \sum_i \alpha_i(t)v_i, \quad \frac{x^2(t) - x}{\|x^2(t) - x\|} = \sum_i \beta_i(t)v_i$$

Using the obvious identity $x^1(t) - x^2(t) = x^1(t) - x + x - x^2(t)$, we can now expand the quantity (3.25): it is thus asymptotic to

$$\frac{1}{C^{1,2}(t)^2} \sum_i \left[C^1(x,t)\alpha_i(t) - C^2(x,t)e^{-(\mu_{x^2}-\mu_{x^1})t}\beta_i(t)\right]^2 \lambda_i$$

Since $C(x^1, x^2, \cdot), C(x^1, x, \cdot), C(x^2, x, \cdot)$ are positive and bounded, $\mu_{x^2} - \mu_{x^1} \geq 0$, and whenever $\lambda_j < 0$ we have $\alpha_j(t), \beta_j(t) \to 0$, the limit for $t \to +\infty$ of (3.25) is positive or 0. Also, at least one of the terms in the sum does not vanish at the limit. If $\mu_{x^1} \neq \mu_{x^2}$, since $\alpha_i(t), \beta_i(t)$ are also bounded and $C^1(x,t)$ does not tend to 0 at infinity, it suffices to notice that $C^1(x,t)\alpha_i(t) \to 0$ for all i implies that $\alpha_i(t) \to 0$ for all i. This is a contradiction with the fact that the α_i are coordinates of a unit vector, and there is at least a nonzero element in the sum: (3.25) is positive in this case. If instead $\mu_{x^1} = \mu_{x^2}$, the proof is a bit

[1] This an application of the standard inequality $a^2 + b^2 \geq 2ab$, where both a and b are positive.

more involved. We start from noticing that, the limit above being 0, it would imply

$$\lim_{t\to+\infty} C^1(x,t)\frac{x^1(t)-x}{\|x^1(t)-x\|} - C^2(x,t)\frac{x^2(t)-x}{\|x^2(t)-x\|} = 0$$

which in turn yields ($\mu := \mu_{x^1} = \mu_{x^2} > 0$)

$$\lim_{t\to+\infty} e^{\mu t}(x^1(t) - x^2(t)) = 0$$

so that $x^1(t) - x^2(t) = o(e^{-\mu t})$, contradiction with the fact that $C^{1,2}(t)$ does not tend to 0. Here as well the value (3.25) is then strictly positive in the limit, and we have proved the lemma. □

A consequence of the proof is that in fact v converges.

Lemma 3.3.5. *$v(\infty)$ is a point.*

Proof. With the notations above,

$$\frac{x^1(t)-x^2(t)}{\|x^1(t)-x^2(t)\|} \sim \frac{1}{C^{1,2}(t)}\left(\sum_i \alpha_i(t)C^1(x,t)v_i - \sum_i \beta_i(t)e^{-(\mu_{x^2}-\mu_{x^1})t}C^2(x,t)(t)v_i\right)$$

The α_i and the β_i clearly admit limits as the quantities $\frac{x^1(t)-x}{\|x^1(t)-x\|}$, $\frac{x^2(t)-x}{\|x^2(t)-x\|}$ tend to a unit vector, since they do so in a Morse chart. For the same reason, $C^1(x,t)$, $C^2(x,t)$ converge to some positive value. For $C^{1,2}(t)$, just remark that the same is true for $\delta(t)$. □

We shall now see how the homothety law (3.21) lets us compute L in terms of the eigenvalues of the hessian matrix. We need the following obvious corollary of (3.21):

Corollary 3.3.6. *Let v_i, v_j be two nonzero eigenvectors of $\mathcal{H}h(x)$, of eigenvalues respectively μ_i and μ_j. Then if $L(v_j) < L(v_i)$ (provided they are both defined) we have $\mu_i < \mu_j$.*

Proof. Without loss of generality we assume we are in a Morse chart so that, in the coordinates given by Morse lemma,

$$\psi^t(x_1, \cdots, x_{2n}) = (e^{-\mu_1 t}x_1, \cdots, e^{-\mu_{2n}t}x_{2n})$$

Since the flow in the chart is linear, (3.21) gives

$$\frac{\|v_j^t\|}{\|v_i^t\|} = e^{-(\mu_j-\mu_i)t}\frac{\|v_j\|}{\|v_i\|} < C\lambda^t \to 0$$

so that necessarily $\mu_j - \mu_i > 0$. □

Remark 3.3.7. *We have already allowed for changes in the Riemannian metric on E around the critical points: we need to highlight this as it will be used several times in what follows.*

We want now to prove the following Lemma, which is going to later imply our main result:

Lemma 3.3.8. *Let x, y, z be three critical points of h, such that there is a negative gradient line connecting x to z, and one connecting y to z. Then if n is odd*

$$L(x-y) \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\mathrm{Ind}_h z}{2} \right\rfloor$$

and if n is even

$$L(x-y) \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{\mathrm{Ind}_h z}{2} \right\rceil$$

where $\mathrm{Ind}_h z$ is the Morse index of h at z.

Proof. We can estimate the maximum possible L for an eigenvector in $T_x W^s(x)$ (remember that we are looking at the negative gradient flow), using the dimension of $E_j(x)$. Let us remark first that function L is defined on every eigenspace: in fact the $E_j(z)$ are stabilised by the linearised flow and are therefore mapped into themselves by the Hessian. Now, the restriction to a subspace of a diagonalisable matrix is still diagonalisable: this fact together with (3.18) shows that there is a basis for $T_z E$ in eigenvectors of the Hessian $\mathcal{H}h(z)$, $v_1, \ldots, v_{2n}$, such that $L(v_{i-1}) \geq L_i$ for all i. Using the homothety law (3.21) we also know that they are ordered by eigenvalue: if $i < j$ then $\mu_i \leq \mu_j$. In particular, if v_i has eigenvalue λ_i we have the relations

$$\lambda_0 \leq \lambda_1 < \cdots < \lambda_{2i} \leq \lambda_{2i+1} < \cdots \leq \lambda_{2n-1}$$

in the case where n is odd, and

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots < \lambda_{2i-1} \leq \lambda_{2i} < \cdots < \lambda_{2n-1}$$

if n is even. We can fill the bases of the different E_j, starting from the highest eigenvalues and from the lowest possible L, $-\left\lfloor \frac{n}{2} \right\rfloor$. We find via elementary calculations that if a vector v is in $T_z W^s(z)$, then it is in $E_j^-(z)$ for $j = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\mathrm{Ind}_h x}{2} \right\rfloor$ whenever n is odd (so that every E_i has dimension 2), or for $j = \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{\mathrm{Ind}_h x}{2} \right\rceil$ if n is even.

Assume now n is odd (the proof for even n being identical). By Theorem 3.2.5, we have the following inequalities, for $t \to +\infty$:

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\mathrm{Ind}_h x}{2} \right\rfloor &\geq L\left(\frac{x^1(t) - x^2(t)}{\|x^1(t) - x^2(t)\|}\right) \geq \\ &\geq L\left(\frac{\psi^{-t}(x^1) - \psi^{-t}(x^2)}{\|\psi^{-t}(x^1) - \psi^{-t}(x^2)\|}\right) \geq L(x-y) \end{aligned}$$

where similarly to the proof of Lemma 3.3.8

$$\lim_{t\to-\infty} x^1(t) = x, \quad \lim_{t\to-\infty} x^2(t) = y, \quad \lim_{t\to+\infty} x^1(t) = \lim_{t\to+\infty} x^1(t) = z$$

□

Remark 3.3.9. *In the proof above we have also allowed for changes in the metric around the critical points. These changes do not affect the validity of Le Calvez's Theorem 3.2.5, and the inequalities above still hold.*

From Section 1.3.2 we know that $CM(h, g; \mathbb{Z})$ is well defined for at least one choice of Riemannian metric g on $\mathbb{R}^{2n}$.

We now define the function

$$I : CM_\bullet(h, g; \mathbb{Z}) \otimes CM_\bullet(h, g; \mathbb{Z}) \to \mathbb{Z}$$

by

$$I(x \otimes y) = \begin{cases} L(x - y) & x \neq y \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\mathrm{Ind}_h x}{2} \right\rfloor & x = y,\ n \text{ odd} \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{\mathrm{Ind}_h x}{2} \right\rceil & x = y,\ n \text{ even} \end{cases} \tag{3.26}$$

on the generators and extend the usual way by

$$I\left(\sum_{i,j} \lambda_{i,j} x_i \otimes x_j\right) := \min_{i,j \mid \lambda_{i,j} \neq 0} I(x_i \otimes x_j) \tag{3.27}$$

Proposition 3.3.10. *The function I as defined in (3.26) induces a filtration on $CM_\bullet(h, g; \mathbb{Z}) \otimes CM_\bullet(h, g; \mathbb{Z})$ for a choice of Riemannian metric g.*

Proof. The differential in the tensor product is defined to be $\partial \otimes \mathrm{Id} + \varepsilon \mathrm{Id} \otimes \partial$, where ∂ is the Morse differential on $CM_\bullet(h, g; \mathbb{Z})$ and ε is a sign (standard definition of the product differential). We may apply the proofs above to the special cases in which either x or y (critical points which negative gradient lines of h flow away from) is equal to z: this shows that $v(\infty)$ also in this case belongs in $T_z W^s(z)$. Switching to $-h$ one can find opposite inequalities in the case in which $x = y$, and a flow line connects y to z. The result is $I(z, z) \geq I(x, y)$ in the former case, and $I(x, y) \leq I(x, z)$ in the latter. This proves that I gives a filtration on $CM_\bullet(h, g; \mathbb{Z}) \otimes CM_\bullet(h, g; \mathbb{Z})$. □

The expression (3.26) is rather awkward, for essentially two reasons. First, even though the actual number on the diagonal may not depend on the generating function, the description does. Second, it is not apparent what kind of topological information $I(x \otimes x)$ bears. Both points are in stark contrast with the situation outside of the diagonal of the tensor product: for $x \neq y$, $L(x - y)$ is a half of the linking number of the orbits associated to x and y, a piece of totally intrinsic information. We aim now to decode the meaning of $I(x \otimes x)$. A deeper analysis will be provided when comparing with the Floer picture, in a later section.

Fix $\varphi \in \mathrm{Ham}_c(\mathbb{R}^2)$ non degenerate, and h as above. By Viterbo's work [72] we know that differences of Morse indices of critical points of h coincide with differences of the Conley-Zehnder indices of the associated orbits of φ.

If $x \in \mathrm{Crit}(h)$, we denote by γ_x the associated periodic orbit of φ. We may therefore normalise the Conley-Zehnder index the following way:

$$\mathrm{Ind}_h(x) - \sigma(h) = CZ(\gamma_x) + 1 \tag{3.28}$$

Recalling that $\sigma(h) = n-1$, we plug this equality and (3.28) in (3.26). What we obtain is the following, more natural definition of I:

$$I(x \otimes y) = \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & x \neq y \\ -\left\lceil \frac{\mathrm{CZ}(\gamma_x)}{2} \right\rceil & x = y \end{cases} \tag{3.29}$$

We are going to justify the normalisation (3.28) later on, and prove that it coincides with one of the usual ones.

3.4 Extension to Viterbo-type generating functions

Fix $\varphi \in \mathrm{Ham}_c(\mathbb{R}^2)$: Laudenbach-Sikorav's Theorem [68] provides a generating function $S : \mathbb{R}^2 \times \mathbb{R}^N \to \mathbb{R}$ which is quadratic at infinity. Such a function is obtained by cutting any Hamiltonian isotopy between the identity and φ into $\mathscr{C}^1$-small components, say n of them. The proof of Sikorav's theorem as developed by Brunella [14] shows that in such a case there exists a generating function defined on $\mathbb{R}^{4n+2}$. We are now going to prove the central theorem, let us restate it:

Theorem 3.4.1. *Let $S : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ be a GFQI representing a non degenerate compactly supported Hamiltonian diffeomorphism $\varphi \in Ham_c(\mathbb{R}^2)$. Then there exists a non degenerate quadratic form on $\mathbb{R}^l$ and a metric g on $\mathbb{R}^{2+k+l}$ such that $(S \oplus Q, g)$ is a Morse-Smale, Palais-Smale pair, and the function*

$$I : CM(S \oplus Q, g; \mathbb{Z}) \otimes CM(S \oplus Q, g; \mathbb{Z}) \to \mathbb{Z}$$

$$I(x \otimes y) = \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & x \neq y \\ -\left\lceil \frac{\mathrm{CZ}(\gamma_x)}{2} \right\rceil & x = y \end{cases}$$

increases along the tensor product differential.

Proof. Fix S as in the statement, and let h be a Le Calvez generating function. Then, by Viterbo Uniqueness Theorem, there exist two non degenerate quadratic forms

$$Q_i : \mathbb{R}^{k_i} \to \mathbb{R}, \;\; i = 1, 2, \;\; k_i \in \mathbb{N}$$

and a gauge equivalence

$$\psi : \mathbb{R}^{2n+k_1} \to \mathbb{R}^{2n+k_1}$$

such that

$$(h \oplus Q_1) \circ \psi = S \oplus Q_2$$

We are clearly going to set $Q := Q_2$ and $l := k_2$. Now, let g be the Riemannian metric on $\mathbb{R}^{2n}$ for which we know that I defines a filtration on $CM(h, g; \mathbb{Z}) \otimes CM(h, g; \mathbb{Z})$. If g' is the associated Riemannian metric defined on $\mathbb{R}^{k+k_1}$ we have the following isomorphisms of complexes

$$\begin{aligned} &CM(h,g;\mathbb{Z}) \cong CM(h \oplus Q_1, g'; \mathbb{Z}) \cong \\ &\cong CM((h \oplus Q_1) \circ \psi, \psi^* g'; \mathbb{Z}) = CM(S \oplus Q, \psi^* g'; \mathbb{Z}) \end{aligned}$$

The filtration I is then pushed forward along these isomorphism of chain complexes, proving the statement of the Theorem. □

3.4.1 Linking filtration and continuation maps

We now analyse the behaviour of the linking filtration under continuation maps, i.e. chain homotopies between Morse complexes induced by regular homotopies between Morse data. Given the result of the previous section about the construction of the filtration for general generating functions, it suffices to understand the behaviour of the filtration under continuation maps between the functions constructed by Le Calvez. Let (h_i, g_i) for $i = 0, 1$ be two such Morse-Smale and Palais-Smale pairs defined on the same vector bundle E over $\mathbb{R}^2$ (in particular, the two quadratic forms at infinity coincide). From Lemma 3.2.8 we know that both

$$CM(h_i, g_i; \mathbb{Z}) \otimes CM(h_i, g_i; \mathbb{Z})$$

carry a filtration I as defined above. We want to compare the filtration on the two sides of a continuation map between the two chain complexes. We consider continuation maps of the kind we described in Chapter 1. We consider a convex homotopy (H, G) between (h_0, g_0) and (h_1, g_1), and let $s \in (-\delta, 1 + \delta)$ be the time of the homotopy. The homotopy H contains a term which only depends on s in its construction: we ignore it, since it does not affect the properties of L as long as the Riemannian metric G is a small perturbation of a product metric.

Because the twist condition read off the generating function is a convex condition on the derivatives, a homotopy as above has the property that any $(H(s, \cdot), G(s; \cdot, \cdot))$ is a Le Calvez pair, so that L has the Lyapunov property for every fixed s, and likewise for any fixed s there exists a dominated splitting as above.

Lemma 3.4.2. *Define* $\tilde{L} : (-\delta, 1 + \delta) \times E \to \mathbb{Z}$ *as*

$$(s, x_0, \dots, x_{2n-1}) \mapsto L(x_0, \dots, x_{2n-1})$$

If for some $\tilde{x} \in (-\delta, 1 + \delta) \times E$ *the quantity* $\tilde{L}(\tilde{x})$ *is not defined, then there is a positive* ε *such that for all* $0 < t < \|\varepsilon\|$*, if* $\tilde{x}^t = \phi^t_{-\nabla_G H} \tilde{x}$*,* $\tilde{L}(\tilde{x}^t)$ *is defined and*

$$\tilde{L}(\tilde{x}^{-t}) < \tilde{L}(\tilde{x}^t)$$

for all $0 < t < \varepsilon$*.*

Proof. This Lemma is in fact a corollary of the proof of Theorem 3.2.5. Assume $\tilde{L}(\tilde{x})$ is not defined: this means that if $\tilde{x} = (s, x)$, $L(x)$ is not defined either. Then by Theorem 3.2.5 there is an $\varepsilon > 0$ such that $x^{-t} \in W_{j^-}$, $x^t \in W_{j^+}$ for all $0 < t < \varepsilon$, $j^- < j^+$. Here x^t is the flow of the vertical vector field $-\nabla_{g^s} h^s$ evaluated at x at time t. Now, the $W_{j^\pm}$ are open in E: for small times then the vertical part of the flow will still be in W_{j^-} in the negative direction, and in W_{j^+} in the positive one.

More precisely, let $\chi^t : \mathbb{R} \times E \to \mathbb{R}$ be the first projection of the flow of $-\nabla_G H$, and $\eta^t : \mathbb{R} \times E \to E$ be the second one. Then by continuity of χ for any small δ_1, for times $0 \leq |t| < \delta$, $|\chi^t(s, x) - s| < \delta_1$; remark now that the transversality condition between the vector field and the boundary between W_{j^-} and W_{j^+} is open, so for small perturbations of the vector field $\nabla_{g^s} h^s$ it is still verified. By continuity for any $\varepsilon > 0$ there is some $\delta > 0$ such that for $s' \in (s - \delta, s + \delta)$

$$\|\mathrm{pr}_2 \nabla_G H(s', x) - \nabla_{g^s} h^s(x)\| < \varepsilon$$

so by the lines above for an arbitrary small ε the transversality condition is satisfied for times which are small enough (possibly smaller of course than the time in Theorem 3.2.5), and we conclude. □

We now want to extend the result of the previous Lemma to say that I increases along continuation maps. Let (H, G) be a cobordism as above. The proof of Lemma 3.3.4 is still valid in this context, since it is simply a dynamical result which does not depend on the form of the Morse function: if $x, y \in \mathrm{Crit}(h_0)$, $z \in \mathrm{Crit}(h_1)$ and there are gradient lines connecting x and y to z, with the notations above

$$v(\infty) \in T_z W^s(z) = \mathbb{R} \oplus E$$

and the function $\tilde{L}$ really computes the L of the vertical projection of $v(\infty)$. Applying exactly the same proof as above (it is necessary to use the dominated splitting of E at z for the Morse flow induced by h_1), we find the inequality:

$$I(x \otimes y) \leq I(z \otimes z)$$

We have thus proved the Proposition:

Proposition 3.4.3. *Let (h_i, g_i) be pairs for which the filtration I is defined on*

$$CM(h_i, g_i; \mathbb{Z}) \otimes CM(h_i, g_i; \mathbb{Z})$$

Assume Φ is a continuation map

$$\Phi : CM(h_0, g_0; \mathbb{Z}) \to CM(h_1, g_1; \mathbb{Z})$$

given by a linear cobordism (H, G). Then

$$I(x \otimes y) \leq I(\Phi(x) \otimes \Phi(y))$$

As a corollary, we obtain that the same property holds true for continuation maps between more general generating functions, not necessarily of Le Calvez-type.

Corollary 3.4.4. *Let* $(S\oplus Q, g), (S'\oplus Q', g')$ *be two pairs as in Theorem 3.4.1, defined on the same vector bundle. Then, if* Φ *is a continuation map induced by a linear cobordism of Morse data,* $\forall x, y \in \mathrm{Crit} S$,

$$I((x,0)\otimes(y,0)) \leq I(\Phi(x,0)\otimes\Phi(y,0))$$

Proof. Use that fact that both S, S' coincide with a function of Le Calvez-type up to stabilisation, then apply the previous proposition. □

We now report a Lemma about the relation between the $\mathscr{C}^0$-metric on generating functions and the Hofer metric. This Lemma is a variation of Theorem 1.2.B in [12], and the author could not find a proof in the literature. A proof may also be given via the theory of Hamilton-Jacobi equations.

Lemma 3.4.5. *Let* $\varepsilon > 0$*. Given* $\varphi \in Ham_c(\mathbb{R}^2)$ *there exists a* $\delta > 0$ *such that for every* $\psi \in Ham_c(\mathbb{R}^2)$ *with* $d_H(\varphi,\psi) < \delta$ *there are GFQI* S *and* T *for* φ *and* ψ *respectively such that*

$$\|S - T\|_\infty \leq \varepsilon \tag{3.30}$$

Proof. By bi-invariance of the Hofer metric, it suffices to check that if H is a compactly supported Hamiltonian generating φ, and $\|H\|_{(1,\infty)} \leq \delta$ for a positive real number δ, then there exists a GFQI S for φ, a non degenerate quadratic function on the fibres Q (which therefore generates the identity) with

$$\|S - Q\|_\infty \leq \varepsilon$$

For any compactly supported Hamiltonian G on $\mathbb{R}^2$, consider an autonomous Hamiltonian L on $\mathbb{R}^2$, constantly equal to δ on the support of G, with very small first and second derivatives, positive and compactly supported. Then ϕ^1_L has a generating function F without fibre variables. The function F is moreover positive and maximum equal to δ: this is true because (in the notation from [73]) $c_-(\phi^1_L) = 0$ by positivity of the Hamiltonian, $c_-(\phi^1_L) = \min F$ by the fact that F has no fibre variable and spectrality axiom, $c_+(\phi^1_L) = \max F$ for the same reason and every fixed point of ϕ^1_L has action either 0 or δ.

Take now a decomposition of ϕ^1_H into N Hamiltonian diffeomorphisms $\phi^1_{H_i}$ for $i = 0, \dots, N-1$, so that for every $t \in [0,1]$

$$\phi^t_{H_i} = \phi_H^{\frac{1+t}{N}}$$

We may thus assume $\phi^1_{H_i}$ to be generated by the Hamiltonian

$$H_i(t,x) := \frac{1}{N} H\left(\frac{i+t}{N}, \phi_H^{\frac{1+t}{N}}(x)\right)$$

For N large enough then:

- Each H_i is $\mathscr{C}^2$-small;
- $\|H_i\|_{(1,\infty)} \leq \delta = \|H\|_\infty$;
- $\|H_i\|_\infty \leq \delta$

Then we apply the above remark to each H_i: we obtain generating functions F_i for L_i which are positive and with maximum less than or equal to C. Now, since H_i is $\mathscr{C}^2$-small, it admits a generating function S_i. Since $H_i \leq L_i$ we moreover have

$$S_i \leq F_i \leq \delta$$

In fact $F_i - S_i$ generates $L_i - H_i$ (because L_i is constant on $\mathrm{Supp}(H_i)$) which is a positive Hamiltonian, and we end as above for the inequality $S_i \leq F_i$. The inequality $F_i \leq \delta$ was already proved above. We then apply composition formulas for generating functions to the S_i (see Lemma 1.3.23): we obtain a generating function quadratic at infinity S, with associated non degenerate quadratic form Q such that

$$\|S - Q\|_\infty \leq N\delta$$

We get what we wanted setting $\delta = \frac{\varepsilon}{N}$. □

We have then the following dynamical application:

Proposition 3.4.6. *Let $\varphi \in Ham_c(\mathbb{R}^2)$ be a non degenerate Hamiltonian diffeomorphism. Assume that the action values of distinct fixed points of φ are distinct. There then exists a $\delta > 0$ such that if ψ is non degenerate and $d_H(\varphi, \psi) < \delta$ and $x \neq y \in \mathrm{Fix}(\varphi)$ are non degenerate critical points,* $\mathrm{lk}(x, y) = k$*, there exist $x', y' \in \mathrm{Fix}(\psi)$ with* $\mathrm{lk}(x', y') = k$.

Proof. Take generating functions h and k for φ and ψ respectively from Lemma 3.4.5, perturb them to have only finitely many critical points, and fix metrics g, g' allowing for the definition of the respective Morse complexes. We may assume that

$$\|h - k\|_\infty < \varepsilon$$

for

$$\varepsilon < \frac{1}{2} \min\{h(x) - h(y) | x, y \in \mathrm{Crit}(h)\}$$

Define the continuation map

$$\Phi : CM(h, g; \mathbb{Z}) \to CM(k, g'; \mathbb{Z})$$

and its opposite Ψ:

$$\Psi : CM(k, g'; \mathbb{Z}) \to CM(h, g; \mathbb{Z})$$

We have to show that, given any critical point x of h corresponding to a non degenerate fixed point of φ, there exist two gradient lines, $\gamma_{\Phi,x}$ and $\gamma_{\Psi,x}$, the former defining Φ, negatively asymptotic to x and positively asymptotic to some x' (which is then a critical point of k), and the latter defining Ψ, negatively asymptotic to x' and positively asymptotic to x. By standard results in Morse theory, $\Psi \circ \Phi$ is homotopic to $\mathrm{Id}_{CM(h,g;\mathbb{Z})}$. Furthermore, let $\lambda = h(x)$. Fix a $\mu' > 0$ such that

$$\mu + 2\varepsilon < \min\{h(x) - h(y) | x, y \in \mathrm{Crit}(h)\}$$

which is possible from the hypothesis in the statement. Then by definition

$$CM^{\lambda+\mu}(h, g; \mathbb{Z}) = CM^{\lambda+\mu+2\varepsilon}(h, g; \mathbb{Z})$$

This implies that, by the above, that

$$\Psi \circ \Phi|_{CM^{\lambda+\mu}(h,g;\mathbb{Z})} \sim \mathrm{Id}_{CM^{\lambda+\mu}(h,g;\mathbb{Z})}$$

because, under the composition $\Psi \circ \Phi$, the filtration may jump upwards by 2ε at most. This is still true at the quotient

$$\Psi \circ \Phi|_{CM^{(\lambda-\mu,\lambda+\mu)}(h,g;\mathbb{Z})} \sim \mathrm{Id}_{CM^{(\lambda-\mu,\lambda+\mu)}(h,g;\mathbb{Z})} \tag{3.31}$$

but in this case the complex $CM^{(\lambda-\mu,\lambda+\mu)}(h, g; \mathbb{Z})$ is generated by exactly one critical point: the composition (3.31) is exactly the identity. This proves that the gradient lines $\gamma_{\Phi,x}$ and $\gamma_{\Psi,x}$ exist. Repeat the argument for y, to find $\gamma_{\Phi,y}$ and $\gamma_{\Psi,y}$. Since these lines exist, we have

$$\begin{aligned} I(x \otimes y) = I(\gamma_{\Phi,x}(-\infty) \otimes \gamma_{\Phi,y}(-\infty)) \le I(\gamma_{\Phi,x}(+\infty) \otimes \gamma_{\Phi,y}(+\infty)) = \\ = I(\gamma_{\Psi,x}(-\infty) \otimes \gamma_{\Psi,y}(-\infty)) \le I(\gamma_{\Psi,x}(+\infty) \otimes \gamma_{\Psi,y}(+\infty)) = I(x \otimes y) \end{aligned}$$

We define then $(x', y') := (\gamma_{\Phi,x}(+\infty), \gamma_{\Phi,y}(+\infty))$. We are now left to show that $x' \neq y'$. By the $\mathscr{C}^0$-Lipschitz property of continuation maps, $|h(x) - k(x')|, |h(y) - k(y')| < \varepsilon$, where ε is the one appearing in the proof of Theorem. Since now by definition of ε we have $|h(x) - h(y)| \geq 2\varepsilon$ by triangle inequality

$$|k(x') - k(y')| \geq |h(x) - h(y)| - |k(x') - h(x)| - |k(y') - h(y)| > 2\varepsilon - 2\varepsilon = 0$$

Since $k(x') \neq k(y')$, obviously $x' \neq y'$. We may now at last conclude that

$$\mathrm{lk}(xy) = I(x \otimes y) = I(x' \otimes y') = \mathrm{lk}(x', y')$$

□

3.5 Floer's picture

Since the Morse theory of generating functions is known to be isomorphic to Floer Homology for (compactly supported) Hamiltonian diffeomorphisms (see for instance [50]) as filtered modules, it may come as no surprise that one may define a filtration I on Floer complexes, again computing the linking number outside the diagonal. The two filtrations, as we are going to see below, coincide under the isomorphism Morse-Floer. We are going to define the Floer filtration in this section, building on work mostly by Hofer-Wysocki-Zehnder [32], Hutchings [34], and Siefring [66].

Normalisation of the Conley-Zehnder index The Conley-Zehnder index is going to play a fundamental role in the very definition of I. In order to compare our results with existing ones, we need to make sure that the normalisation we use for the index coincides with the reference literature. Remember that we had set

$$\mathrm{Ind}_h(x) - \sigma(h) = CZ(\gamma_x) + 1$$

in (3.28). The previously cited works normalise the index the following way: if $H : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathscr{C}^2$-small Hamiltonian, and α is a 1-periodic orbit of the Hamiltonian diffeomorphism of $\mathbb{R}^2$ defined by H and associated to a critical point α of H (under the $\mathscr{C}^2$-smallness hypotheses, fixed points of the time 1-map are exactly the critical points of H), we set

$$\mathrm{CZ}(\alpha) + 1 = \mathrm{Ind}_H(\alpha) \tag{3.32}$$

We have to check that the two normalisations (3.28) and (3.32) coincide on a simple example.

Consider H defined as

$$H(x,y) = \frac{\varepsilon}{2}\rho(x,y)(x^2+y^2)$$

where ρ is a plateau function we use to make H compactly supported. For small ε the Hamiltonian H is indeed $\mathscr{C}^2$-small, and the origin is a non degenerate fixed point of the time 1 map φ. The Morse index of 0 for H is clearly 0, so that by (3.32)

$$CZ(0) = -1$$

We now have to find a generating function for H of the Le Calvez kind and compute its Morse index at the point representing the origin of the plane as a fixed point. The problem is local in nature, so ignore the contribution of ρ, and we assume H generates a genuine rotation of angle ε. A generating function for the symplectomorphism $R^{-1} \circ \varphi$ (remember, R is the clockwise rotation of order 4) is

$$h_0(x_0,x_1) = \frac{x_0^2}{2}\frac{\sin\varepsilon}{\cos\varepsilon} + \frac{1}{\cos\varepsilon}x_0x_1 + \frac{x_1^2}{2}\frac{\sin\varepsilon}{\cos\varepsilon}$$

so that using Le Calvez's results we obtain the following generating function:

$$h(x_0,x_1,x_2,x_3) = \frac{x_0^2}{2}\frac{\sin\varepsilon}{\cos\varepsilon} + \frac{1}{\cos\varepsilon}x_0x_1 + \frac{x_1^2}{2}\frac{\sin\varepsilon}{\cos\varepsilon} - x_1x_2 + x - 2x_3 - x_3x_0$$

The critical point corresponding to the origin of the plane is the origin of $\mathbb{R}^4$: we have to compute the Morse index of h there. The Hessian of h at $0 \in \mathbb{R}^4$ is

$$M = \begin{pmatrix} \frac{\sin\varepsilon}{\cos\varepsilon} & \frac{1}{\cos\varepsilon} & 0 & -1 \\ \frac{1}{\cos\varepsilon} & \frac{\sin\varepsilon}{\cos\varepsilon} & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

and we approximate it by means of the obvious Taylor expansion:

$$M' = \begin{pmatrix} \frac{\varepsilon}{1-\varepsilon^2/2} & \frac{1}{1-\varepsilon^2/2} & 0 & -1 \\ \frac{1}{1-\varepsilon^2/2} & \frac{\varepsilon}{1-\varepsilon^2/2} & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{3.33}$$

The signatures of M and M' coincide. We compute then the determinants of the square submatrices of M' whose diagonal is contained in the one of M': for small $\varepsilon > 0$, the top-left entry

$$\frac{\varepsilon}{1-\varepsilon^2/2}$$

is positive, and the determinants of the other submatrices are negative. There is exactly one sign change for these determinants, which implies

$$\sigma(M') = \sigma(M') = 1$$

By definition then

$$\mathrm{Ind}_h(0) = 1$$

From (3.28) and knowing that $\sigma(h) = 1$ we find

$$CZ(\gamma_0) = -1$$

which is what we wanted to show.

3.5.1 A filtration on the Floer complex

Consider again $\varphi \in \mathrm{Ham}_c(\Sigma)$ generated by a Hamiltonian H. If φ is non degenerate, for a generic choice of almost complex structure J the Floer complex $(CF(H, J; \mathbb{Z}), \partial)$ is well defined. Let $x, y \in \mathrm{Fix}(\varphi)$. We may define the function

$$I(x \otimes y) = \begin{cases} \frac{1}{2}\mathrm{lk}(x, y) & x \neq y \\ -\left\lceil \frac{\mathrm{CZ}(x)}{2} \right\rceil & x = y \end{cases} \tag{3.34}$$

on the product of generators in complete analogy to what was done above, and we extend to the whole complex $CF(H, J; \mathbb{Z}) \otimes CF(H, J; \mathbb{Z})$ as done in (3.27). In this case it induces a filtration as well: we have to prove that

Proposition 3.5.1.

$$I(\partial(x \otimes y)) \geq I(x \otimes y)$$

In this last expression, ∂ denotes the tensor product differential

$$\partial(x \otimes y) := \partial x \otimes y + (-1)^{CZ(x)} x \otimes \partial y$$

Before starting with the proof of the proposition, we mention the following equality:

Lemma 3.5.2. *In the notation of the previous chapter, if x is a 1-periodic orbit of φ,*

$$I(x \otimes x) = \alpha_-^\tau(x) \tag{3.35}$$

for a trivialisation τ, canonical up to homotopy.

Proof. First off, τ is constructed taking a capping disc u in Σ for the orbit x. Since the pullback bundle $u^*T\Sigma$ has contractible basis, it is globally trivial and any two trivialisations are homotopic. Adding to this fact our assumption that $\pi_2(\Sigma) = 0$, we obtain that τ is indeed canonical up to homotopy. After remarking this the Lemma is a trivial consequence of Equation (1.21), because p takes values in $\{0, 1\}$. □

We are going to use the canonical trivialisation τ of the above Lemma throughout this section.

Proof. Of Proposition3.5.1 In the Hamiltonian Floer setting, we are lead to count intersections between pseudoholomorphic curves defining the differential, in order to estimate the variation of I. Let $u, v : \mathbb{R}_s \times \mathbb{S}^1_t \to \Sigma$ be two Floer cylinders with distinct images. The two cylinders have therefore only finitely many intersections: since the images are distinct, the asymptotic description given by Siefring in [67] implies that the cylinders do not intersect in a neighbourhood of infinity, but they cannot have infinitely many intersections in a compact set, else they would have the same image. Let $u(\pm\infty, \cdot) = x_\pm$, $v(\pm\infty, \cdot) = y_\pm$. Here we make the abuse of notation of giving the same name to a fixed point and the periodic orbit through that fixed point. Denote by $\bar{u}$ and $\bar{v}$ the graphs of u and v respectively. Remark that we are allowing for the case in which the cylinders are trivially constant at a Hamiltonian orbit.

Looking at the (unordered) pair (u, v) as a homotopy between the braids composed by the asymptotic Hamiltonian orbits, we easily see from (1.2) that to each (transverse) intersection counted in $\bar{u} \cdot \bar{v}$ corresponds a linking increase of 2 since such intersections are positive. If $x_\pm \neq y_\pm$ then we have the following formula:

$$\bar{u} * \bar{v} = \bar{u} \cdot \bar{v} + \iota_\infty(\bar{u}, \bar{v}) = \bar{u} \cdot \bar{v} = I(x_+ \otimes y_+) - I(x_- \otimes y_-)$$

In the second equality we use $x_\pm \neq y_\pm$ to deduce that $\iota_\infty(\bar{u}, \bar{v}) = 0$, and in the third equality we use this assumption again to infer that

$$I(x_\pm \otimes y_\pm) = \frac{1}{2}\mathrm{lk}(x_\pm, y_\pm)$$

Assume now that $x_- = y_-$ (the case at $+\infty$ being analogous). Then the same calculations as above lead to

$$\bar{u} * \bar{v} = \bar{u} \cdot \bar{v} + \iota^\tau_{-\infty}(\bar{u}, \bar{v}) - \alpha^\tau_+(x_-) = I(x_+, y_+) - I(x_-, y_-)$$

since $\bar{u}\cdot\bar{v} + \iota^\tau_{-\infty}(\bar{u}, \bar{v}) = \frac{1}{2}\mathrm{lk}(x_+, y_+)$ by (1.2). These are all the possible cases we have to consider: in the tensor product differential we have to count constant cylinders at periodic orbits and Floer cylinders connecting periodic orbits of Conley-Zehnder index difference 1. □

Floer theory comes with a package of higher operations, see for instance [64]. We now give some results in the case of pair-of-pants products. We first have to describe the geometric meaning of the intersections of the relevant curves.

A pair of pants with p legs (and one waist) is a branched p-fold cover of the cylinder $\mathbb{R}_s \times \mathbb{S}^1_t$, trivial on the region $s < 0$ and non trivial on $s > 0$. The point $(s,t) = (0,0)$ is the only branching point. The preimage under the covering map of sets $\{ s = s_0 \}$ are circles of length p for $s_0 < 0$, and the disjoint union of p circles of length 1 for $s_0 > 0$. Considering two pairs of pants $u, v : S_p \to \Sigma$ (not necessarily holomorphic), and denote by u_s, v_s the restrictions

$$u_s, v_s : \pi^{-1}(\{s\} \times \mathbb{S}^1) \to \Sigma$$

with $\pi : S_p \to \mathbb{R} \times \mathbb{S}^1$ being the branched covering map. Since intersections of pseudoholomorphic curves are isolated, we may assume that the graphs of u_s and v_s do not intersect for small $|s| > 0$, but we allow for intersections in $\pi^{-1}(\{0\} \times \mathbb{S}^1)$. Then, if without loss of generality $s > 0$, the pair (u_{-s}, v_{-s}) forms p braids with two strands of length 1, and (u_s, v_s) forms one braid with two strands, and of length p. The relation between these two sets of braids is easily described by the following Lemma:

Lemma 3.5.3. *In the same context as above, for $\varepsilon > 0$ sufficiently small we have*

$$\frac{1}{2}\mathrm{lk}(u_\varepsilon, v_\varepsilon) = \frac{1}{2}\sum_{i=1}^{p} \mathrm{lk}(u^i_{-\varepsilon}, v^i_{-\varepsilon}) + \theta(u,v) \tag{3.36}$$

where $\theta(u, v)$ is the count of intersections between the graphs of u and v in $\pi^{-1}(\{0\} \times \mathbb{S}^1)$.

Proof. Denote as above u_ε and v_ε are the restrictions of u and v respectively to the unique connected component of $\pi^{-1}(\{\varepsilon\} \times \mathbb{S}^1)$. By continuity of u and v, since $(0,0)$ is the branching point of the cover $S_p \to \mathbb{R} \times \mathbb{S}^1$, there exists a numbering $u^j_{-\varepsilon}$ (resp. $v^j_{-\varepsilon}$) for the maps $\mathbb{S}^1 \to \Sigma$ defined by restriction of u (resp. v) to the connected components of $\pi^{-1}(\{-\varepsilon\} \times \mathbb{S}^1)$ so that for all $j \in \{0, \dots, p-1\}$ and $t \in [0,1]$ we have

$$\lim_{\varepsilon \to 0} u^j_{-\varepsilon}(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t+j), \ \lim_{\varepsilon \to 0} v^j_{-\varepsilon}(t) = \lim_{\varepsilon \to 0} v_\varepsilon(t+j)$$

Consider the two equalities for $t = 0$: when passing through the branching point (letting $\varepsilon \to 0$) we are concatenating the p braids of two strands and length 1 to obtain one braid with two strands and length p. Up to a small perturbation of either pair of pants, we may assume that each intersection counting towards $\theta(u, v)$ is away from the branching point and transverse. Each transverse intersection appearing in $\pi^{-1}((-\varepsilon, 0)) \times \mathbb{S}^1)$ makes the linking number of the corresponding braid change by 2. Each transverse intersection in $\pi^{-1}((0, \varepsilon)) \times \mathbb{S}^1)$ makes the linking number of (u_s, v_s) change by 2, for $s \in (0, \varepsilon)$. For each transverse intersection in $\pi^{-1}(\{0\} \times \mathbb{S}^1)$ the two points of view are equivalent. Since lk is additive under concatenation, this proves the Lemma. ☐

Remark 3.5.4. *Remark that the order of concatenation of the p braids is not canonical, but the two sides of (3.36) do not depend on it.*

We may now define an analogue of I on the tensor power of $CF(H, J; \mathbb{Z})$:

$$I_p : CF(H, J; \mathbb{Z})^{\otimes p} \otimes CF(H, J; \mathbb{Z})^{\otimes p} \to \mathbb{Z}, \tag{3.37}$$

$$I_p((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) := \sum_{i=1}^{p} I(x_i^1 \otimes x_i^2) \tag{3.38}$$

By definition of the tensor product differential, I_p is still a filtration (we are counting, as above, intersections between trivial Floer cylinders and Floer cylinders appearing in the definition of the differential). Let $H^{\#p}$ be the concatenation of the Hamiltonian H, so that $H^{\#p}$ generates ϕ_H^p. The Floer complex of $H^{\#p}$ also bears a linking filtration I, defined as before in this chapter. Let

$$\mathcal{P}_p : CF(H, J; \mathbb{Z})^{\otimes p} \to CF(H^{\#p}, J_p; \mathbb{Z}) \tag{3.39}$$

be the p-pair of pants product on the Floer complex. We want to show:

Theorem 3.5.5. *The pair of pants product (3.39) preserves the filtrations I and I_p, i.e.*

$$I_p((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) \leq I(\mathcal{P}_p(x_1^1 \otimes \cdots \otimes x_p^1) \otimes \mathcal{P}_p(x_1^2 \otimes \cdots \otimes x_p^2))$$

Notation 3.5.6. *If $u : S_p \to \Sigma$ is a pair of pants in Σ, we denote by $\bar{u}$ its graph*

$$\bar{u} : S_p \to S_p \times \Sigma$$

Assume u^i is a pair of pant $u^i : S_p \to \Sigma$, negatively asymptotic to $x_1^i, \dots, x_p^i$ and positively asymptotic to y^i, for $i = 1, 2$. If $u^1 = u^2$, then $y^1 = y^2 = y$ and for all j, $x_j^1 = x_j^2 = x_j$, so that we can estimate the difference

$$I(y \otimes y) - I_p((x_1 \otimes \cdots \otimes x_p) \otimes (x_1 \otimes \cdots \otimes x_p))$$

using the Conley-Zehnder indices only, assuming that the transversality is achieved. If the two pairs of pants are distinct instead, we are going to need to compute the Siefring product $\overline{u^1} * \overline{u^2}$ between the two graphs

$$\overline{u^1}, \overline{u^2} : S_p \to S_p \times \Sigma$$

and remark that in this case $\overline{u^1} * \overline{u^2} \geq 0$ to conclude.

For the next Lemma we are not yet requiring the moduli spaces to be cut-out transversely, we just ask for the existence of these curves. Recall that for the pairs of pants product we use the setup provided by [62] and [25].

Lemma 3.5.7. *Let H be a Hamiltonian on Σ, φ be its time 1-map. Let $x_1^1, \dots, x_p^1, x_1^2, \dots, x_p^2$ be fixed points of φ, and y^1, y^2 be fixed points of φ^p. Assume that u^i is a pair of pants appearing in the computation of $\mathcal{P}(x_1^i \otimes \cdots \otimes x_p^i)$ with positive asymptotic orbit $y^i \in \mathrm{Fix}(\varphi^p)$ and that $\mathrm{Im}(u^1) \neq \mathrm{Im}(u^2)$. Then*

$$\overline{u^1} * \overline{u^2} \leq I(y^1 \otimes y^2) - I((x_1^1 \otimes \cdots x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) \tag{3.40}$$

Proof. Remark that because intersections between pseudoholomorphic curves are isolated and an asymptotic description of the behaviour of pseudoholomorphic half-cylinders (see [67]) if $\mathrm{Im}(u^1) \neq \mathrm{Im}(u^2)$ then there are only finitely many intersections.

Suppose that $x_j^1 \neq x_j^2$ for $j \in \{1, \dots, k\}$, and that for different indices $x_j^1 = x_j^2 =: x_j$. Either $y^1 = y^2 =: y$ or not: we carry out the proof assuming the equality, the other one being essentially the same. This is a straightforward computation using (1.25) and the definitions of $\Omega_\pm^\tau$. In particular the reader should be mindful that if $x_j^1 \neq x_j^2$ then by definition $\Omega_-^\tau(x_j^1, x_j^2) = 0$.

$$I(y \otimes y) - \sum_{j=1}^{p} I(x_j^1 \otimes x_j^2) \geq \alpha_-^\tau(y) - \frac{1}{2}\sum_{j=1}^{k} \mathrm{lk}(x_j^1, x_j^2) - \sum_{j=k+1}^{p} \alpha_-^\tau(x_j) =$$

$$= -\Omega_+^\tau(y) - \frac{1}{2}\sum_{j=1}^{k} \mathrm{lk}(x_j^1, x_j^2) - \sum_{j=k+1}^{p} \Omega_-^\tau(x_j, x_j) = \iota_\infty^\tau(u^1, +\infty; u^2, +\infty) -$$

$$-\Omega_+^\tau(y) - \iota_\infty^\tau(u^1, +\infty; u^2, +\infty) - \frac{1}{2}\sum_{j=1}^{k} \mathrm{lk}(x_j^1, x_j^2) - \sum_{j=1}^{p} \Omega_-^\tau(x_j, x_j) =$$

Now, recall that by (1.2) and Lemma 3.5.3 we have

$$-\iota_\infty^\tau(u^1, +\infty; u^2, +\infty) - \frac{1}{2}\sum_{j=1}^{k} \mathrm{lk}(x_j^1, x_j^2) - \sum_{j=k+1}^{p} \iota_\infty^\tau(u^1, z_j; u^2, z_j) = \overline{u^1} \cdot \overline{u^2}$$

so that, summing and subtracting $\sum_{j=k+1}^{p} \iota_\infty^\tau(u^1, z_j; u^2, z_j)$ in the equality above we find

$$I(y \otimes y) - \sum_{j=1}^{p} I(x_j^1 \otimes x_j^2) \geq \overline{u^1} \cdot \overline{u^2} + \big(\iota_\infty^\tau(u^1, +\infty; u^2, +\infty) - \Omega_+^\tau(y, y)\big) +$$

$$+ \left(\sum_{j=1}^{p} \iota^\tau(u^1, z_j; u^2, z_j) - \Omega_-^\tau(x_j^1, x_j^1)\right) = \overline{u^1} \cdot \overline{u^2} + \iota_\infty(u^1, u^2) = \overline{u^1} * \overline{u^2}$$

which is what we wanted. □

Remark 3.5.8. *We remark that in the Lemma we only have an inequality and not an equality as soon as for some $x_j := x_j^1 = x_j^2$ we have $\alpha_-^\tau(x_j) < \alpha_+^\tau(x_j)$: this may happen depending on the parity of the Conley-Zehnder index of x_j.*

Lemma 3.5.9. *In the setup of the previous Lemma,*

$$0 \leq I(y^1 \otimes y^2) - I((x_1^1 \otimes \cdots x_p^1) \otimes (x_1^2 \otimes \cdots \otimes x_p^2)) \tag{3.41}$$

Proof. Clearly it suffices to prove that the left hand side of (3.40) is positive or 0, and in turn it is enough to show that the $\overline{u^i}$ are holomorphic for a certain choice of almost complex structure on $S_p \times \Sigma$. Let u represent either pair of pants u^i. Recall that by Lemma (1.3.35) the map

$$(\pi, u) : S_p \to \mathbb{R} \times \mathbb{S}^1 \times \Sigma$$

is holomorphic for a choice of almost complex structure $\tilde{J}$ on $\mathbb{R} \times \mathbb{S}^1 \times \Sigma$. We have a map

$$(\pi, \mathrm{Id}) : S_p \times \Sigma \to \mathbb{R} \times \mathbb{S}^1 \times \Sigma$$

satisfying $(\pi, \mathrm{Id}) \circ \bar{u} = (\pi, u)$. Denote by $\mathbf{0}$ the branching point of π: the branched cover (π, Id) is a local biholomorphism on $(S_p \setminus \{\mathbf{0}\} \times \Sigma)$ and may therefore be used to pull $\tilde{J}$ back to an almost complex structure on $(S_p \setminus \{\mathbf{0}\} \times \Sigma)$, which we denote by $\overline{J}$. Since the curve u is holomorphic near the branching point (the Hamiltonian terms vanish outside the cylinder-like ends), the almost complex structure considered by Faber on $\mathbb{R} \times \mathbb{S}^1 \times \Sigma$ is the product almost complex structure (see the proof of [25, Lemma 2.3]). This implies that the almost complex structure $\overline{J}$ may be extended to a smooth almost complex structure on $S_p \times \Sigma$, since (π, Id) is holomorphic between the product structures on

$$\mathbb{R} \times \mathbb{S}^1 \times \Sigma \text{ and } S_p \times \Sigma$$

by definition. By this argument, we obtain $\bar{J}$-holomorphicity for both $\overline{u^1}$ and $\overline{u^2}$. All intersections in $\overline{u^1} \cdot \overline{u^2}$ will be positive. The term $\iota_\infty(\overline{u^1}, \overline{u^2})$ is positive or zero by holomorphicity as well. The Siefring product $\overline{u^1} \cdot \overline{u^2}$ is positive or zero, and by (3.40) we have proved the Lemma. □

For the following Lemma we assume that the transversality one needs for the good definition of the product is achieved, to restrict our attention to pair of pants of null Fredholm index.

Lemma 3.5.10. *The following inequality is verified:*

$$p - 1 \leq I(y^1 \otimes y^1) - I((x_1^1 \otimes \cdots \otimes x_p^1) \otimes (x_1^1 \otimes \cdots \otimes x_p^1)) \tag{3.42}$$

Proof. To prove this lemma, we may just simply apply the definition using the Conley-Zehnder index translation given by the product:

$$CZ^\tau(y) = \sum_{j=1}^{p} CZ^\tau(x_j) + 2(2 - p - 1)$$

Divide by 2 and take the floor function: given the elementary inequality

$$\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil \;\; \forall x, y \in \mathbb{R}$$

we obtain

$$\left\lceil \frac{CZ^\tau(y)}{2} \right\rceil \leq \sum_{j=1}^{p} \left\lceil \frac{CZ^\tau(x_j)}{2} \right\rceil + 1 - p$$

and this yields the claimed result. □

Now Theorem 3.5.5 follows easily from Lemmata 3.5.9 and 3.5.10.

3.6 Relations to other work and future directions

We finish this chapter mentioning the relations between these results and works of others. The dynamical Corollary 3.4.6 is in fact a weaker but more elementary version of [3, Theorem 2]. They prove that given a collection of k orbits of a Hamiltonian diffeomorphism φ of an aspherical surface $\Sigma_{g,p}$ (if $p \geq 1$ they assume the diffeomorphism to be compactly supported) which draws a braid $b \in \mathcal{B}_{k,g,p}$, if one perturbs φ to ψ with a small Hofer deformation, there are k orbits of ψ of braid type b. Essentially, they prove that the pseudoholomorphic curves in the continuation maps do not intersect, thus providing a braid isotopy. Some generalisation of Alves and Meiwes's result have already appeared: see [35] and [4].

Another fundamental work we would like to mention is the one by Connery-Grigg [19]. He used the machinery provided by Hofer-Wysocki-Zehnder [32] and Siefring [67],[66] to deeply analyse the geometry of the Floer complex. On the one hand he manages to construct singular foliations obtained from the Floer differential (this is connected to Le Calvez's work on transverse foliations), and on the other he finds a topological characterisation of the generators of the Floer complex representing the fundamental class, generalising previous work by Humilière, Le Roux and Seyfaddini [33]. Most of the technical lemmata in [19, Section 3.2.1] may be cast in the language of generating functions, and the proofs adapted with ease within the Morse-theoretical context we provide, building on Le Calvez results.

We wish to mention as well that a related linking filtration exists in Embedded Contact Homology and the related theory of PFH. It measures linking number with respect to a fixed Reeb orbit. So far, to the best of the author's knowledge, it has found a few applications: see for instance the paper where Hutchings introduced it [36] and other works by Nelson and Weiler [55][54]. In particular, Hutchings's result has been re-proved by Le Calvez using generating functions, see [43] and its recent generalisation, due to Bramham and Pirnapasov [13].

The natural question arising from the above connections is then to what extent generating functions may be used to re-prove these results obtained via various flavours of Floer theory, and whether they can offer a way to write simpler proofs (for known and new theorems), at least in the context of low dimensional dynamics, since they may be defined for Hamiltonian diffeomorphisms on any surface (classical for surfaces with genus, for the sphere see [2]).

In some ongoing work, not advanced enough to be reported here, we aim however to exploit the interaction between the filtration I_p and the higher operation in Floer homology to prove results about the topology of periodic orbits with larger periods.

Appendix A

The Filtration I in the Morse-Bott case

In this Appendix we indicate what the self-linking number of a transversely non degenerate fixed point should be. A Hamiltonian diffeomorphism which presents only transversely non degenerate critical points will have a Morse-Bott generating function. The Conley-Zehnder index of a transversely non degenerate critical point is to be defined, and its place has to be taken by the Robbin-Salamon index. First we compute the asymptotic values of L the same way as it was done in Section 3.3. The methods are identical, but we need to keep track of the nullity of the Hessian. After this, to give a topological meaning to the number we find, we rewrite it as a function of the Robbin-Salamon index of the orbit. To do this, after recalling the definition of Robbin-Salamon index, we prove that the difference of Morse-Bott indices coincides with the difference of Robbin-Salamon indices.

A.1 I in the Morse-Bott case

Definition A.1.1. *Let M be a smooth manifold. A function $f \in \mathscr{C}^\infty(M;\mathbb{R})$ is said to be Morse-Bott if* $\mathrm{Crit}(f)$ *is a disjoint union of smooth closed submanifolds (B_j) of M, and the Hessian is transversely non degenerate, i.e.*

$$\forall j,\, \forall x \in B_j,\, \ker \mathcal{H}f(x) = T_x B_j$$

Let $\mathrm{Ind}(B)$ *denote the signature of the Hessian on a complement of $T_x B_j$ in $T_x M$. The Morse-Bott index of B_j is then defined to be*

$$MB(B_j) = \mathrm{Ind}(B_j) + \frac{\dim(B_j)}{2}$$

If $x \in B_j$, we also write $MB(x) := MB(B_j)$, and since the Morse index of the Hessian is locally constant on critical submanifolds, this definition is good by connectedness of B_j.

Remark A.1.2. *The fact that the dimension of the critical submanifold is divided by 2 in the definition of the Morse-Bott index depends on the fact that signatures appearing in the computation of the Robbin-Salamon index, later defined, at the endpoints of a Lagrangian path are also divided by 2. This definition we give here is nonstandard.*

Suppose that a generating function $h : E \to \mathbb{R}$ for $\varphi \in \mathrm{Ham}_c(\mathbb{R}^2)$ constructed as in [44, Section 1.12] is Morse-Bott on the interior of its support. The same proof that implies that L is well defined at the difference of a pair of critical points implies that it is continuous on a difference of points on critical submanifolds.

Lemma A.1.3. *Let B be a critical submanifold for h. Then L is continuous on $B \times B \setminus \Delta$. It is in particular constant on its connected components.*

Proof. Let $x \neq y$ be two distinct points in B, and assume by contradiction that $L(x-y)$ is not defined. By Theorem 3.2.5 there is then some $\varepsilon > 0$ such that for all $|t| < \varepsilon$, $L(x^t - y^t)$ is defined. But x and y belong to the critical submanifold B, and $x^t = x$, $y^t = y$ for all real times, contradiction. □

Our aim is to associate a number I to a transversely non degenerate fixed point of φ.

Definition A.1.4. *Let $\varphi \in Ham_c(\mathbb{R}^2)$, and $x \in \mathrm{Fix}(\varphi)$. The fixed point x is transversely non degenerate if for any V complement of* $\mathrm{Span}\,\dot{x}(0)$ *in $\mathbb{R}^2$, $d_x\varphi|_V$ does not have 1 as eigenvalue.*

The critical point of h associated to a transversely non degenerate fixed point will belong in a critical submanifold of h. Let $\mathcal{C}(h)$ be the family of connected critical submanifolds of h. We want to state a new definition for I, holding in the case of Morse-Bott functions:

$$I : \mathcal{C}(h) \times \mathcal{C}(h) \to \mathbb{Z}$$

$$(B, C) \mapsto L(x-y) \quad \text{for any } x \in B, y \in C, x \neq y$$

This is a good definition by the lemma above. We are going to prove that an inequality as in Lemma 3.3.8 still holds.

We can repeat the proof of Lemma 3.3.4 almost word for word, to find

Lemma A.1.5. *Given a pair of gradient lines converging to a point $z \in B$, defining the function v as in Lemma 3.3.4, we have $v(\infty) \in T_zW^s(z) \subseteq TW^s(B)$.*

Proof. Repeat the same argument as in Lemma 3.3.4 in Morse-Bott charts. One only needs to check that $v(\infty) \notin T_zB$: this is achieved just remarking that $\alpha_i(t), \beta_i(t) \to 0$ when $\lambda_i = 0$ as well. □

Now, as a consequence again of Proposition 3.3.1, any vector in TB has a lower value of L compared to any vector in $TW^s(B)$: this motivates the following definition. Remember that $n \in \mathbb{N}$ here refers to the length of the decomposition of φ as a product of twist maps.

$$I : \mathcal{C}(h) \times \mathcal{C}(h) \to \mathbb{Z},\ (B,C) \mapsto \begin{cases} L(B,C) & B \neq C \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{MB(B)+\frac{\dim B}{2}}{2} \right\rfloor & B = C,\ 2 \not| n \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{MB(B)+\frac{\dim B}{2}}{2} \right\rceil & B = C,\ 2|n \end{cases}$$

Obviously this definition generalises the previous one, and we have the following property.

Definition A.1.6. *Let A, B, C be three connected critical manifolds for a generating function h as above, not necessarily distinct, and assume that there is a gradient line negatively asymptotic to A and positively asymptotic to B. Then:*

$$I(A,C) \leq I(B,C)$$

A.2 Morse-Bott and Robbin-Salamon indices

A.2.1 The Robbin-Salamon index

Let us recall the definition of Robbin-Salamon index from [63]. It is a generalisation of the Maslov index for loop, whose definition may for instance be found in [72, Section 1].

Let Λ be a Lagrangian in $\mathbb{R}^{2n}$, $t \mapsto \Lambda(t)$ a path of Lagrangians such that $\Lambda(0) = \Lambda \in Gr(Lag(\mathbb{R}^{2n}))$. We denote by $Gr(Lag(\mathbb{R}^{2n}))$ the Lagrangian Grassmannian in $\mathbb{R}^{2n}$. Then

Proposition A.2.1 (Robbin, Salamon [63])**.** *Let W be a Lagrangian complement of Λ in $\mathbb{R}^{2n}$, $v \in \Lambda$ and $t \mapsto w(t) \in W$ defined for small t so that $v + w(t) \in \Lambda(t)$. Then the form*

$$Q(\Lambda, \dot{\Lambda}(0))(v) := \frac{d}{dt}_{\restriction t=0} \omega(v, w(t))$$

is independent of the choice of W. Moreover, it is invariant under application of a linear symplectomorphism to Λ.

Now, for a fixed Lagrangian subspace V of $\mathbb{R}^{2n}$ we may define its Maslov cycle $\Sigma(V)$: it is the set of Lagrangian subspaces intersection V non trivially. The set $\Sigma(V)$ is in fact stratified by the dimension of the intersections. Assume now again that $t \mapsto \Lambda(t)$ is a path in Lagrangian subspaces. At every t such that $\Lambda(t) \in \Sigma(V)$ we define the crossing form

$$\Gamma(\Lambda, V, t) := Q(\Lambda, \dot{\Lambda}(t))|_{\Lambda(t) \cap V} \tag{A.1}$$

A crossing is called regular if $\Gamma(\Lambda, V, t)$ is non degenerate. It is shown in [63, Lemma 2.2] that any Lagrangian path is homotopic rel endpoints to a path with only regular crossings.

Definition A.2.2. *Assume Λ has only regular crossings. The Robbin-Salamon index for the path Λ, defined on $[a, b]$, with respect to the subspace V as*

$$RS(\Lambda, V) := \frac{1}{2}\mathrm{sign}\Gamma(\Lambda, V, a) + \sum_{t|\Lambda(t)\in\Sigma(V)} \mathrm{sign}\Gamma(\Lambda, V, t) + \frac{1}{2}\mathrm{sign}\Gamma(\Lambda, V, b)$$

Remark A.2.3. *Warning: sign here denotes the difference between the number of negative and the number of positive eigenvalues*[1] *of a Gram matrix for Q.*

The Robbin-Salamon index has several powerful properties, most notably:

(*Homotopy*) Two paths of Lagrangian subspaces are homotopic with fixed endpoints if and only if they have the same Robbin-Salamon index.

(*Catenation*) If the path of Lagrangian subspaces $\Lambda : t \mapsto \Lambda_t$ is defined on an non empty interval $[a, b]$ and $a < c < b$, then

$$RS(\Lambda, V) = RS\left(\Lambda|_{[a,c]}, V\right) + RS\left(\Lambda_{[c,b]}, V\right)$$

(*Zero*) If the path of Lagrangian subspaces $t \mapsto \Lambda_t$ is defined on an non empty interval $[a, b]$ and the dimension of the intersection $\Lambda_t \cap V$ is constant, then $RS(\Lambda, V)$=0.

(*Localisation*) If $V = \mathbb{R}^n \times \{0\}$ and $\Lambda(t) = Graph(A(t))$ is Lagrangian, $t \in [a, b]$, then

$$RS(\Lambda, V) = \frac{1}{2}\mathrm{sign}\Gamma(A(b)) - \frac{1}{2}\mathrm{sign}\Gamma(A(a))$$

We refer to [63] for an exhaustive discussion of these and other properties of RS.

Definition A.2.4. *The Robbin-Salamon index of a path of linear symplectomorphisms $\psi : t \mapsto \psi$ is defined to be*

$$RS(\psi) := RS(Graph(\psi), \Delta)$$

where Δ is the diagonal of $\mathbb{R}^{2n+2n}$ equipped with the symplectic form $-\omega \oplus \omega$.

If ψ is the linearisation at a non degenerate fixed point of a Hamiltonian isotopy, we recover the Conley-Zehnder index.

[1] This is the opposite of the definition in [63], we adopt it for consistency with the above results.

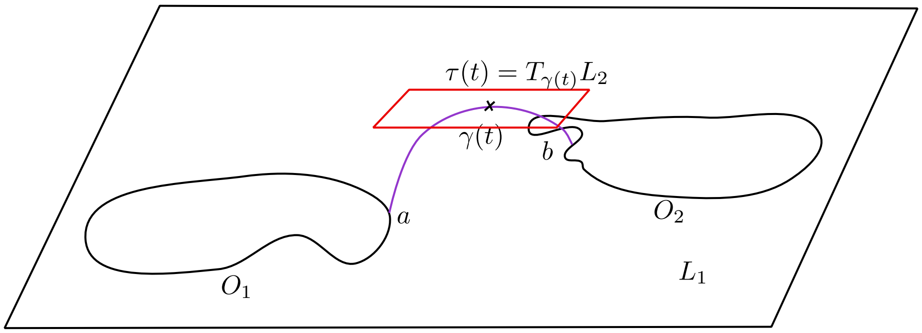


Figure A.1: The path γ.

A.2.2 The differences of indices coincide

We want to prove the following.

Proposition A.2.5. *Let h be a Morse-Bott generating function for a (therefore) transversely non degenerate Hamiltonian diffeomorphism $\varphi \in Ham_c(\mathbb{R}^{2n}, \omega)$. Given critical points $x, y \in \mathrm{Crit}(h)$ associated to orbits γ_x and γ_y, one has*

$$RS(d_{\gamma_x}\varphi) - RS(d_{\gamma_y}\varphi) = MB(x) - MB(y)$$

To do so, we repeat the construction in [72, Section 4] adapting it to this more general context using the Robbin-Salamon index. Let $h : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ a Morse-Bott generating function for a compactly supported Hamiltonian diffeomorphism φ of $\mathbb{R}^{2n}$. Denote by $L_1 = 0_{T^*(\mathbb{R}^{2n}\times\mathbb{R}^k)}$ and $L_2 = \mathrm{Gr}(dh)$: they are both Lagrangians in $T^*(\mathbb{R}^{2n} \times \mathbb{R}^k)$, and the Lagrangian generated by h is the image of L_2 under symplectic reduction over the coisotropic submanifold

$$W := T^*\mathbb{R}^{2n} \times \mathbb{R}^N \times \{0\}$$

Let O_1, O_2 be two critical manifolds of h: they intersections, still denoted by the same name, between L_1 and L_2. Assume that O_1 and O_2 are either points (corresponding to non degenerate fixed points of Hamiltonian diffeomorphisms) or circles (corresponding to $\mathbb{S}^1$-families of transversely non degenerate fixed points)[2]. Take any $a \in O_1$, $b \in O_2$, $a \neq b$, and consider a curve γ contained in L_2, $\gamma(0) = a$, $\gamma(1) = b$. The Lagrangian path

$$\tau(t) = T_{\gamma_1(t)}L_1 \;\; \forall t \in [0, 1]$$

defines a map $\tau : [0, 1] \to Gr(Lag(\mathbb{C}^{N+2n}))$. See Figure A.1 for a depiction of γ and τ.

Define then

$$m'(O_1, O_2) = RS(\tau, \mathbb{R}^{2n+N}) \in \frac{1}{2}\mathbb{Z}$$

[2]The following proofs however still work without assumptions on the dimension of O_1 and O_2

By (Catenation) and (Zero) axioms the index $m'(O_1, O_2)$ does not depend on the choice of the points a, b on the intersections, let us see it for a (case of b being identical). If a' is another point in O_1, we may connect it to a by a path in O_1. Along this path, the tangent spaces are constantly equal to $\mathbb{R}^{N+2n}$, and therefore the path has 0 Robbin-Salamon index by (Zero). By (Catenation) now the index of the chosen path coincides with that of γ. We also show that the number $m'(O_1, O_2)$ does not depend on the choice of γ either. In fact, every two γ are homotopic through an ambient homotopy because $\pi_1(\mathbb{R}^{2n}) = 0$, and this ambient homotopy provides us with a homotopy between the Lagrangian paths: by the (Homotopy) axioms, the Robbin-Salamon indices coincide.

We are going to prove now that on one side

$$m'(O_1, O_2) = MB(O_1) - MB(O_2)$$

and on the other that

$$m'(O_1, O_2) = RS(d_{\gamma_2}\varphi) - RS(d_{\gamma_1}\varphi)$$

where γ_i is the orbit (non degenerate or transversally non degenerate) associated to the O_i. Let us prove the former equality.

We need the following

Proposition A.2.6. *Let $W := T^*\mathbb{R}^{2n} \times \mathbb{R}^N \times \{0\}$, τ be as above, and τ' be the reduction of τ along W. Then*

$$RS(\tau, \mathbb{R}^{N+2n}) = RS(\tau', \mathbb{R}^{2n})$$

To show it we are going to apply to Lemma A.2.7 (which we shall prove shortly). This will allow us to reduce the problem to the simpler case in which the generating function has no fibre variables. Before proving Lemma A.2.7, let us show that to prove stability under reduction of the Robbin-Salamon index of τ we just need to verify stability at one endpoint of the path at most.

Assume, to start, that O_1 and O_2 are points and the intersections are transverse: this is the case in which h is Morse. Then by (Zero) and (Catenation) axioms $m'(O_1, O_2)$ is equal to the Maslov index of a loop of Lagrangians: consider a path $\gamma_1 : [0, 1] \to L_1$ with $\gamma_1(0) = b$ and $\gamma_1(1) = a$. We can use it to define a Lagrangian path τ_1 with the properties that

$$\tau_1(0) = T_b L_2,\ \tau_1(1) = T_a L_2,\ \tau_1(t) \pitchfork T_{\gamma_1(t)} L_1$$

Let μ denote the usual Maslov index (a definition may be found in [72, Section 1]): then

$$RS(\tau, \mathbb{R}^{2n+N}) = RS(\tau \# \tau_1, \mathbb{R}^{2n+N}) = \mu(\tau \# \tau_1) \tag{A.2}$$

where we used the (Catenation) and (Zero) axioms, together with the fact that the Robbin-Salamon index recovers the Maslov index in the case of loops. and [72, Proposition 3] implies our Proposition A.2.6. We now drop the Morse assumption.

The second easiest case is found when both intersections, O_1 and O_2, have the same dimension. In such a case we may repeat the argument we just used: define a Lagrangian path τ_1 such that $\tau\#\tau_1$ is a loop and the intersection $\tau_1(t)\cap\mathbb{R}^{2n+N}$ has constant dimension (for the existence of this loop we use [63, Corollary 4.4]). This still implies (A.2), and as before [72, Proposition 3] proves Proposition A.2.6.

Assume now that O_1 is a transverse intersection, and O_2 is a circular Morse-Bott intersection (the roles are symmetrical and the argument may be repeated swapping them). Construct a Lagrangian path τ_1 such that

$$\tau_1(0)=T_bL_2,\,\forall t\in(0,1]\tau_1(t)\pitchfork\mathbb{R}^{2n+N}),\;\tau_1(1)=T_aL_2$$

Such a path may be constructed by [63, Corollary 4.4], together with a result the fact that the set of Lagrangian subspaces with 1-dimensional intersection with a fixed one has positive codimension. We apply once again [72, Proposition 5], finding

$$RS(\tau\#\tau_1,\mathbb{R}^{2n+N})=\mu(\tau\#\tau_1)$$

The path τ_1 now however has a crossing at time 0, which contributes by $\pm\frac{1}{2}$ to the total Robbin-Salamon index. Denote by Q the intersection form of τ_1 at time 0. We obtain

$$RS(\tau,\mathbb{R}^{2n+N})+\frac{1}{2}\mathrm{sign}Q=\mu(\tau\#\tau_1)$$

Since the right hand side of the equation is known to be stable under symplectic reduction by [72, Proposition 3], to prove Proposition A.2.6 it suffices to check that sign$Q$ also is.

Lemma A.2.7. *Let $L_1=0_{T^*(\mathbb{R}^{2n}\times\mathbb{R}^k)}$, $L_2:=Graph(dh)$ be Lagrangian submanifolds in $T^*(\mathbb{R}^{2n}\times\mathbb{R}^k)$ and τ a Lagrangian path as above. Assume that L_1 and L_2 intersect in a circle C. Let $b\in C$, Q denote the crossing form at b for T_bL_2 with respect to T_bL_1. Let W denote the coisotropic submanifold of $T^*(\mathbb{R}^{2n}\times\mathbb{R}^k)$ given by*

$$W:=T^*\mathbb{R}^{2n}\times\mathbb{R}^N\times\{0\}$$

Then

$$\mathrm{sign}\Gamma\left(\tau,\mathbb{R}^{2n+N},1\right)=\mathrm{sign}\Gamma\left(\tau',\mathbb{R}^{2n},1\right)$$

where τ' is the Lagrangian path obtained from τ via symplectic reduction over W.

Proof. Clearly the integrable orthogonal distribution at the intersection point is given, in the chosen trivialisation, by $W^\perp(b)=\{0\}\times\{0\}\times\mathbb{R}^N\times\{0\}$.

We first prove that we can identify the intersection with its reduction. Since $\mathbb{R}^{N+2n}\subset T_bW$, we have (the apostrophe denoting the reduction):

$$(\mathbb{R}^{N+2n}\cap T_bL_2)':=(\mathbb{R}^{N+2n}\cap T_bL_2\cap T_bW)+W^\perp(b)/W^\perp(b)=$$
$$=(\mathbb{R}^{N+2n}\cap T_bL_2)+\{0\}\times\{0\}\times\mathbb{R}^N\times\{0\}/\{0\}\times 0\times\mathbb{R}^N\times 0$$

Since now $T_bL_1 \cap T_bL_2 \subset \mathbb{R}^{2n} \times \{0\} \times \{0\} \times \{0\}$, then

$$T_bL_1 \cap T_bL_2 \cap (\{0\} \times \{0\} \times \mathbb{R}^N \times \{0\}) = \{0\}$$

We can conclude that

$$(T_bL_1 \cap T_bL_2)' \cong T_bL_1 \cap T_bL_2 \tag{A.3}$$

The dimension of the intersection of the tangent spaces being 1, we need to check that the two signatures coincide. The reduction of L_1, L_1' is the zero section of $T^*\mathbb{R}^{2n}$, while the one of L_2, L_2' is the Lagrangian that h generates. We want to compute the signature of the quadratic form associated to the intersection between $T_{b'}L_1'$ and the horizontal Lagrangian $T_{b'}M = T_bL_2'$. Choose a Lagrangian complement V' for T_bL_2' in $T_{b'}T^*\mathbb{R}^{2n}$: it extends to $V := V' \times V_1$, a Lagrangian complement for T_bL_2. The space V_1 is then a Lagrangian complement for the second factor of T_bL_2 with respect to the splitting

$$T_b(T^*(\mathbb{R}^{2n} \times \mathbb{R}^k)) = T_{\mathrm{pr}_1 b}T^*\mathbb{R}^{2n} \oplus T_{\mathrm{pr}_2 b}T^*\mathbb{R}^N$$

Now everything is set: consider (notations as above) the path of Lagrangian spaces τ'. For any $v' \in T_{b'}L_1' \cap T_{b'}L_2'$ there is then a unique $w'(t) \in V'$ such that $v' + w'(t) \in \tau'(t)$. Remark that by the isomorphism (A.3) the preimage v of v' in $T_bL_1 \cap T_bL_2$ is well defined. Then if ω' is the symplectic form on the reduction $T^*\mathbb{R}^{2n}$, by naturality (pull-back of the reduced symplectic form coinciding with the restriction on the coisotropic manifold) for any preimage of $w'(t)$, say $w(t) \in V$,

$$\Gamma\left(\tau', \mathbb{R}^{2n}, 1\right)(v') = \frac{d}{dt}_{\restriction t=1} \omega'(v', w'(t)) = \frac{d}{dt}_{\restriction t=1} \omega(v, w(t))$$

The right-hand side is in fact the crossing form $\Gamma\left(\tau, \mathbb{R}^{N+2n}, 1\right)(v)$: while the vector $w(t)$ a priori does not realise $v + w(t) \in \tau(t)$, if $\bar{w}(t)$ does satisfy $v + \bar{w}(t) \in \tau(t)$, then remark that v is $\omega-$orthogonal to $w(t) - \bar{w}(t)$. The symplectic form ω is in fact the direct sum of the canonical forms on $T^*\mathbb{R}^{2n}$ and $T^*\mathbb{R}^N$, $w(t) - \bar{w}(t)$ belongs in the space $\{0\} \times \{0\} \times \mathbb{R}^N \times \mathbb{R}^N$ (their difference is reduced to 0 under reduction) and $v \in T_b(T^*\mathbb{R}^{2n}) \times \{0\} \times \{0\}$. □

Remark A.2.8. *Clearly, if T_bL_1 and T_bL_2 have intersection of dimension different from 1, the result of the Lemma above is still true because of (A.3).*

We have finished the proof of Proposition A.2.6 as well. We are now reduced to prove the equality in the case in which the generating function has in fact no fibre variables. From now on $N = 0$, and L_2 is now a graph in $T^*\mathbb{R}^{2n}$. Define the path τ as above: we have to compute

$$RS(\tau, \mathbb{R}^{2n})$$

Clearly now, if $\mathcal{H}h(\gamma(t))$ is the Hessian of h at $\gamma(t)$,

$$T_{\gamma(t)}L_2 \cap \mathbb{R}^{2n} = T_{\gamma(t)}Graph(dh) \cap \mathbb{R}^{2n} \neq \{0\} \Leftrightarrow \det \mathcal{H}h(\gamma(t)) = 0$$

The count is going to be exactly the same as in the Morse case, since intersections correspond to times in which the Hessian becomes degenerate: the intersection is positive if an eigenvalue from positive becomes negative, and negative in the opposite case, by [63, Theorem 1.1 (2)][3]. This count yields

$$RS(\tau, \mathbb{R}^{2n}) = MB(b) - MB(a) \tag{A.4}$$

The path τ up to homotopy rel endpoints has in fact only regular crossings: this implies that the signs of the eigenvalues which vanish at a crossing time need to change, and also that if the dimensions of O_1 and O_2 are different the change in the dimension of the intersection happens at the beginning or at the end of the path. Remark that we are not assuming that the eigenvalues change sign one at a time: this would mean requiring the crossings to be simple, which is a stronger condition.

Remark A.2.9. *From the proof above we see the reason why the Robbin-Salamon index is defined this way (with the dimension of the critical manifold to be divided by 2): this condition is necessary for Equation (A.4) to hold, since the eigenvalues dropping to 0 at the extremities contribute for a half.*

We now show that

$$m'(O_1, O_2) = RS(d_{\gamma_1}\varphi) - RS(d_{\gamma_2}\varphi)$$

To achieve it, we make use of perturbations defined in [17] and of the previous point to prove the result without fibre variables: let us suppose for instance that both O_1 and O_2 are both Morse-Bott circles (the other cases are similar but easier). The function h then represents a transversely non degenerate Hamiltonian orbits for a certain Hamiltonian diffeomorphism φ generated by a Hamiltonian H. Then we can perturb locally H using a small Morse function on the circle with exactly one maximum and one minimum: the perturbed Hamiltonian H_δ generates a non degenerate diffeomorphism, and the two orbits are decomposed into two couples of non degenerate orbits, each term of the couple corresponding to either the maximum or the minimum of the Morse function. Let a and b be respectively on O_1 and O_2 be chosen in such a way that after the perturbation (i.e. Hamiltonian deformation of L_2) they both correspond to the minimum of the Morse function. Let $L_{2,\delta}$ be the perturbed Lagrangian: then $a, b \in L_{2,\delta} \cap L_1$. Now, by the result of Viterbo in [72] and using his notation,

$$m(a, b, L_1, L_{2,\delta}) = CZ(\gamma_{2,\delta}) - CZ(\gamma_{1,\delta})$$

where $\gamma_{i,\delta}$ are the perturbed orbits, and m is computed using a path on the perturbed Lagrangian (it is the Maslov index of a Lagrangian loop, whose construction we mimicked here). These indices correspond to the Robbin-Salamon indices by [63, Remark 5.4]

$$CZ(\gamma_{i,\delta}) = RS(d_{\gamma_{i,\delta}}\varphi), \ i = 1, 2$$

[3]Some care needs to be taken when comparing conventions about the signature. In [63], the signature is defined as the difference between the number of positive and the number of negative eigenvalues, here we are taking the opposite convention.

Let τ_δ be the path we use to compute this index.

We apply [17, Proposition 3.2] to have

$$RS(d_{\gamma_2}\varphi) - RS(d_{\gamma_1}\varphi) = CZ(\gamma_{2,\delta}) - CZ(\gamma_{1,\delta})$$

The homotopy $s \mapsto H_{s\delta}$, for $s \in [0,1]$, chosen a path between a and b contained in L_2, gives then a homotopy with fixed endpoints between the Lagrangian path τ we have already defined, and the concatenation of three paths, the first τ^a joining T_aL_2 to $T_aL_{2,\delta}$, the one in the middle being τ_δ, and the third one the inverse of τ^b joining T_bL_2 to $T_bL_{2,\delta}$. We use [17, Proposition 3.2] again to see that

$$RS(\tau^a, \mathbb{R}^{2n}) = RS(\tau^b, \mathbb{R}^{2n})$$

These indices in fact only depend on the Morse indices of the critical points of the perturbation corresponding to the orbits $\gamma_{i,\delta}$. We can now apply the (Catenation) and the (Homotopy) axioms to conclude that

$$m'(O_1, O_2) = CZ(\gamma_{2,\delta}) - CZ(\gamma_{1,\delta})$$

This ends the proof in the case for O_1 and O_2 both circles.

To cover the one where only O_1 is a circle, perturb only O_1 and repeat the same argument (of course, in this case we are not going to have cancellations). This ends the proof of Proposition A.2.5.

A.2.3 I and the Robbin-Salamon index

The results in the previous section imply that we can again normalise the Robbin-Salamon index in a way that it corresponds to the Morse-Bott index as we did for the Morse and Conley-Zehnder indices in (3.28). We set then, for a Morse-Bott generating function h of a diffeomorphism φ with connected compact critical submanifold B and $x \in B$

$$MB(x) - \sigma(h) = RS(\gamma_x) + 1 \tag{A.5}$$

Remark that if the dimension of B is 1, i.e. it is a circle, $\gamma_x(0)$ is a part of an $\mathbb{S}^1$-family of fixed points (even if φ is not autonomous), whereas if γ_x is isolated and $x \in B$, then $\dim B = 0$. We can then recover the dimension of the critical submanifold purely by dynamical information. Replacing MB in the definition of I using (A.5) we can define

$$I : \mathrm{Fix}(\varphi) \times \mathrm{Fix}(\varphi) \to \mathbb{Z},\, I : (\gamma_x, \gamma_y) \mapsto \begin{cases} \frac{1}{2}\mathrm{lk}(\gamma_x, \gamma_y) & \gamma_x \neq \gamma_y \\ -\left\lceil \frac{RS(\gamma_x) + \frac{\dim B}{2}}{2} \right\rceil & \gamma_x = \gamma_y, x \in B \end{cases}$$

This formula is consistent with the Morse case.

Appendix B

A Generalisation of Khanevsky's Proof

In this section we are going to prove Theorem 2.2.6 in the restricted case where $k = 2$ with the tools contained in Section 1.3.3. The methods here will also generalise verbatim to higher numbers of strands, but to diffeomorphisms of certain braid types only.

M. Khanevsky in [41] proved that given a non displaceable disc contained in an annulus, the Hofer norm of a diffeomorphism fixing such a disc (not necessarily pointwise) grows at least linearly with the absolute value of the rotation number of the diffeomorphism. As we see in this section, Khanevsky's proof may be generalised in an elementary way to our setting: we are going to consider a two disjoint discs in $\mathbb{D}$ whose boundaries constitute a premonotone lagrangian configuration $\underline{L}$, and give lower bounds on the Hofer norm of an element in $\mathrm{Ham}_{\underline{L}}(\mathbb{D}, \omega)$ using the linking number of the induced braid.

Let $\mathbb{D} \subset \mathbb{C}$ be the two dimensional unit open disc, with area form ω normalised such that $\int_{\mathbb{D}} \omega = 1$. Let D_1, D_2 be two open disjoint discs in $\mathbb{D}$ such that neither is displaceable in the complement (the area condition reads $A = Area(D_i) \in \left(\frac{1}{3}, \frac{1}{2}\right)$). Let $L_i := \partial D_i$, and $\underline{L} = L_1 \times L_2$, and assume moreover that $\varphi(D_i) = D_i$. Remember that for a compactly supported Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}_{\underline{L}}(\mathbb{D}, \omega)$ we can define its braid type, let us call it $b(\varphi, \underline{L}) \in \mathcal{B}_2$. Since $\mathcal{B}_2$ is isomorphic to $\mathbb{Z}$ via the linking number, we identify $b(\varphi, \underline{L})$ with $\frac{1}{2} \cdot \mathrm{lk}(b(\varphi, \underline{L}))$ (the factor of 2 is needed since in our definition of linking number agrees with the group-theoretic one, and the elementary loop $t \mapsto \exp(2\pi i t)$ has linking 2 with the origin of the complex plane). Since $\varphi(D_i) = D_i$, $\frac{1}{2} \cdot \mathrm{lk}(b(\varphi, \underline{L})) \in \mathbb{Z}$.

In a similar way as one can find in [41], we define the set

$$S_n = \left\{ \varphi \in \mathrm{Ham}_{\underline{L}}(\mathbb{D}) \;\middle|\; \varphi(D_i) = D_i,\, \frac{1}{2}\mathrm{lk}(b(\varphi, \underline{L})) = n \right\}$$

and if $\hat{\varphi}$ is a Hamiltonian diffeomorphism such that $\hat{\varphi} \in S_1$, we have a decomposition $S_n = \hat{\varphi}^n S_0$. We want to prove that if $\varphi \in S_n$ then its Hofer norm satisfies

$$\|\varphi\| \geq O(n)$$

and to do this, we show that there is a quasimorphism which is Hofer-Lipschitz, strictly positive on $\hat{\varphi}$ and zero on S_0.

Consider polar coordinates on the disc, so that the symplectic form reads $\omega = \frac{1}{\pi} r\, dr \wedge d\theta$. Up to conjugation by a symplectic diffeomorphism (which does not change the computations of Hofer norms) we can suppose that the picture is symmetric: the two discs D_i are found in the halfplanes of positive and negative x. Let $\hat{\varphi}$ be a compactly supported smooth approximation of a rotation of 2π, and a generating Hamiltonian is for instance

$$H(r, \theta) = \rho(r^2) r^2$$

where ρ is a plateau function which is 1 outside a small neighbourhood of 1 (ρ will in general depend on the area of D_i, and we choose it in a way that $(r, \theta) \mapsto \rho(r^2)$ is equal to 1 in a neighbourhood of $D_1 \cup D_2$). Let j_s be a symplectic embedding from $\mathbb{D}$ into a sphere of area $1+s$, which we denote with $\mathbb{S}^2(1+s)$. It induces a Hofer 1-Lipschitz injection $j_{s*} : \mathrm{Ham}_c(\mathbb{D}) \to \mathrm{Ham}(\mathbb{S}(1+s))$ given by extension by 0 of the Hamiltonians. Remark that the configuration $\underline{L}_s := j_s(\underline{L})$ is monotone in $\mathbb{S}^2$; let η_s be its monotonicity constant that appears as an η in Equation 1.17.

The quasimorphism on $\mathrm{Ham}_c(\mathbb{D})$ of our choice is

$$Q_2 := (\mu^1_{2,\eta_0} \circ j_{0\,*} + \mathrm{Cal}) - \left(\mu^{3A}_{2,\eta_{3A-1}} \circ j_{(3A-1)\,*} + \frac{1}{3A}\mathrm{Cal}\right)$$

Its Hofer Lipschitz constant is 2 (as sum of two Hofer 1-Lipschitz functionals) and we may see as follows that they are furthermore $\mathscr{C}^0$-continuous by the criterion by Entov-Polterovich-Py (in [23], see also [21]). We apply the (Support Control) property and the fact that we may choose the discs we use to compute the quasimorphisms $\mu^a_{k,\eta}$ to see that Q_2, when applied to any diffeomorphism supported on a disc of area less than A, vanishes: an application of the criterion above concludes the argument.

Write $Q_2 = \mu'_0 - \mu'_{3A-1}$, where

$$\mu'_s = \mu^{1+s}_{2,\eta_s} \circ j_{s\,*} + \frac{1}{1+s}\mathrm{Cal}$$

Let us compute $Q_2(H)$: for each of the μ'_s by the (Invariance) property we can use a configuration of two circles, both contained in $j_s(\mathbb{D})$ and whose centres coincide with $j_s(0)$. The area requirement translates to the condition that the radii associated to the two circles are $r = \sqrt{A}$ and $r = \sqrt{1+s-A}$, and . Then by (Lagrangian Control), after mean-normalising H, we find:

$$\mu'_s(\hat{\varphi}) = \frac{s-1}{2}$$

so that

$$Q_2(\hat{\varphi}) = -\frac{1}{2} - \frac{3A-1-1}{2} = \frac{1-3A}{2}$$

Following the steps of [41], and assuming that Q_2 does vanish on S_0 (we are going to later show that this is indeed the case), let us prove that if $\varphi = \hat{\varphi}^n \psi$, then $Q_2(\varphi) = Q(\hat{\varphi}^n) = nQ(\hat{\varphi})$. We have $Q_2(\varphi) = Q(\hat{\varphi}^n) + D(\varphi)$, where $D(\varphi) \in \mathbb{R}$ is bounded in modulus by the defect of the quasimorphism Q (in particular, it is bounded uniformly on φ). Since Q_2 is homogeneous, for all positive integers k

$$Q_2(\varphi) = \frac{Q_2(\varphi^k)}{k} = \frac{Q_2(\hat{\varphi}^{nk}) + D(\varphi^k)}{k} = Q_2(\hat{\varphi}^n) + \frac{D(\varphi^k)}{k}$$

and taking the limit shows that $Q_2(\varphi) = nQ_2(\hat{\varphi})$ (recall that $D(\varphi^k)$ is bounded on k). As Q_2 is Hofer 2-Lipschitz, we conclude that

$$\|\varphi\| \geq |\frac{nQ_2(\hat{\varphi})}{2}| \geq \frac{3A-1}{4}n$$

Remark B.0.1. *We immediately see that our lower bound is an increasing function of A, which is zero in the limit case in which $A = \frac{1}{3}$: when the discs are displaceable, we cannot say anything this way about the Hofer norm of the diffeomorphisms. This is to be expected: for $k = 2$, by [46] (see Remark following his Question 5) if the two discs are displaceable there exists a family of hamiltonian diffeomorphisms, whose Hamiltonians are supported away from one of the two discs, with bounded Hofer norm but whose braid type may be arbitrarily linked.*

It is left to check that Q_2 vanishes on S_0. Using the (Hofer Lipschitz) and ($\mathscr{C}^0-$continuity) properties of Q_2, it suffices to prove that $Q_2(S_0') = 0$, where S_0' is defined as the subgroup of Hamiltonian diffeomorphisms that induce the identity on a neighbourhood of $\partial\mathbb{D} \cup \partial D_1 \cup \partial D_2$ ([41], Lemma 2). The argument is essentially the same as in [41], adapted to our quasimorphism Q_2. To compute Q_2, by (Invariance), this time we choose the obvious configuration $L_i = \partial D_i$. Now, if $\varphi \in S_0'$, it can be decomposed into $\varphi = \varphi_{D_1} \circ \varphi_{D_2} \circ \varphi_P$, where φ_{D_i} is supported on D_i and φ_P on $P = \mathbb{D} \setminus (D_1 \cup D_2)$.

Since the boundaries of the D_i are connected, $\varphi_{D_i}|_{D_i} \in \mathrm{Ham}_c(D_i) \subseteq \mathrm{Ham}_c(\mathbb{S}^2 \setminus (L_1 \cup L_2))$, and by (Lagrangian Control) $Q_2(\varphi_{D_i}) = 0$, thus it is left to show that $Q_2(\varphi_P) = 0$.

To prove this fact, as mentioned in [41] (see also [26]), it is possible to show that $\pi_0(\mathrm{Symp}_c(P)) = \mathbb{Z}^3$, and that it is generated by three Dehn twists around the three boundary components: it is our task to show that if we apply an arbitrarily small Hofer deformation of φ, taking place in $\mathrm{Ham}_c(\mathbb{D})$, we can force $\varphi_P|_P$ to in the connected component of the identity of $\mathrm{Symp}_c(P)$. A Dehn twist around an embedded circle in $\mathbb{D}$ can be represented by a Hamiltonian whose Hofer norm is dependent on the diameter of the normal neighbourhood of the circle that we choose. In particular, we may represent the Dehn twists components in $[\varphi_P|_P] \in \pi_0(\mathrm{Symp}_c(P))$ corresponding to curves near the boundaries of

D_i using $\psi \in \mathrm{Ham}_c(\mathbb{D})$, with $\|\psi\| < \varepsilon$; as for the component corresponding to a curve near $\partial\mathbb{D}$, it cannot appear in the decomposition since $\varphi_P|_P$ is the restriction of an element in S_0' (the presence of such factor forces the linking number to be k, where k is the number of these Dehn twists counted with signs). As a result, $\psi^{-1} \circ \varphi_P|_P$ is in the connected component of the identity in $\mathrm{Symp}_c(P)$, and it is enough to prove that $Q_2(\psi^{-1} \circ \varphi_P|_P) = 0$ by Hofer Lipschitz property of Q_2.

$\psi^{-1} \circ \varphi_P|_P$ is not necessarily in $\mathrm{Ham}_c(P)$, which would be enough to conclude; it may be however deformed to an element in $\mathrm{Ham}_c(P)$ without changing the value of Q_2. Let us consider the Flux homomorphism

$$\widetilde{\mathrm{Symp}_c}(P) \to H^1_c(P; \mathbb{R})$$

(definition and properties may be found in [49]) and let us compute it on two Hamiltonian isotopies defined by K_i where:

- K_i are supported on a small neighbourhood of D_i, equal to 1 in small neighbourhoods of L_i (here the L_i constitute the premonotone lagrangian configuration we had at the beginning of the section, it is not the pair of concentric circles on $\mathbb{S}^2$ we are using to compute the quasimorphism);
- $\mathrm{supp}(K_1) \cap \mathrm{supp}(K_2) = \emptyset$;
- $\mathrm{supp}(K_i) \cap \mathrm{supp}(\psi^{-1} \circ \varphi_P|_P) = \emptyset$ for $i = 1, 2$ and all real times s;

Since the two images $\mathrm{Flux}(\varphi^t_{K_i})$ are linearly independent (they are zero on one generator of $H_1(P, \partial P; \mathbb{R})$ each) and $\dim(H^1_c(P; \mathbb{R})) = 2$ there exist two real times $s_i \in \mathbb{R}$ such that

$$\mathrm{Flux}(\varphi^{s_1}_{K_1}|_P \circ \varphi^{s_2}_{K_2}|_P \circ (\psi^{-1} \circ \varphi_P|_P)) = 0$$

and $\varphi^{s_1}_{K_1} \circ \varphi^{s_2}_{K_2} \circ (\psi^{-1} \circ \varphi_P)$ may be represented by a Hamiltonian supported in P. Moreover,

$$Q_2(\varphi^{s_1}_{K_1} \circ \varphi^{s_2}_{K_2} \circ (\psi^{-1} \circ \varphi_P)) = Q_2(\psi^{-1} \circ \varphi_P)$$

Indeed, since all the supports are disjoint, the three diffeomorphisms commute, therefore Q_2 is additive on them and

$$Q_2(\varphi^{s_i}_{K_i}) = \frac{1}{4}s_i - \frac{1}{4}s_i = 0$$

Now we finish by remarking that

$$0 = Q_2(\varphi^{s_1}_{K_1} \circ \varphi^{s_2}_{K_2} \circ (\psi^{-1} \circ \varphi_P)) = Q_2(\psi^{-1} \circ \varphi_P)$$

by (Support Control) property used on the premonotone lagrangian configuration $L_1 \times L_2$.

Summing up, given a diffeomorphism in S_0, it can be Hofer and $\mathscr{C}^0$-deformed to an element in S_0', changing by an arbitrarily small amount the value of Q_2.

We then showed that in fact $Q_2(S'_0) = 0$, since any symplectic diffeomorphism in S'_0 is Hofer-close to one whose restriction to P is symplectically isotopic to the identity of P. After this small perturbation, we applied a deformation without changing the value of Q_2, arriving to a Hamiltonian diffeomorphism with compact support in P, whose Q_2 value is necessarily 0 by an explicit computation.

Remark B.0.2. *It is possible to compare our estimates with the ones Khanevsky gives in [41]: if n is the rotation number of a non displaceable disc in an annulus of area 1 under an isotopy between the identity and $\varphi \in Ham_c(\mathbb{S}^1 \times (0,1))$, he shows that*

$$\|\varphi\| \geq \frac{2A-1}{2}|n|$$

To be able to compare the two estimates, one should remember that we always assumed our unit disc to have area 1, whereas Khanevsky only assumes the area of the annulus to be 1. In terms closer to our setup, this means that Khanevsky finds such estimate (up to a small change due to area normalisation) for Hamiltonian diffeomorphisms of the disc which may be represented by Hamiltonians supported away from a smaller disc, without further restrictions on the area of the latter disc. We may see then that we weaken the restriction on the support of the Hamiltonians in a considerable way, but we have to ask that the two small discs have the same area (and our estimate is less good than the one given by Khanevsky).

Remark B.0.3. *This proof generalises to a higher number of strands, but will not help to compute the Hofer norm of a diffeomorphism with arbitrary braid type. Let $\varphi \in Ham_{\underline{L}}(\mathbb{D}, \omega)$: for the proof above to hold we would need $b(\varphi)$ to be juxtaposition of braids $\sigma_1 \cdots \sigma_p$, such that there exist p disjoint discs $(D_i)_{i=1,\dots,p}$ in $\mathbb{D}$, satisfying the following properties:*

- *each D_i contains k_i circles in $\underline{L}$, $(L_{i_j})_{j=1,\dots,k_i}$;*
- *$\varphi|_{D_i} = D_i$, and is there conjugated to a rotation;*
- *The circles in $(L_{i_1}, \dots, L_{i_{k_i}})$ realise the sub-braid σ_i.*

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Tresses en dynamique hamiltonienne de basse dimension

Mots clés : Tresses, systèmes hamiltoniens, fonctions génératrices, théorie de Floer, géométrie d'Hofer

Résumé : Dans cette thèse nous étudions les systèmes hamiltoniens en nous appuyant sur la topologie de leurs orbites fermées. Les résultats présentés portent, d'un côté, sur des propriétés des fonctions génératrices associées à un difféomorphisme hamiltonien particulier, et de l'autre sur la distance d'Hofer entre deux difféomorphismes qui réalisent des tresses de types différents.
Dans le premier contexte, on s'appuiera sur des résultats de Patrice Le Calvez pour montrer que toute fonction génératrice d'un difféomorphisme hamiltonien à support compact du plan (à stabilisation près) admet une filtration en enlacements dans la deuxième puissance tensorielle du complexe de Morse. Par “filtration en enlacements” nous entendons une filtration qui a toute paire de points critiques associe un nombre entier, et telle que lorsque les deux points sont distincts la valeur associée est exactement le nombre d'enlacement des deux orbites correspondantes aux points critiques. Il est possible de définir une telle filtration aussi dans le cadre de la théorie de Floer hamiltonienne, et d'étudier son comportement par rapport au produit en homologie. Les résultats de l'auteur dans cette direction n'ont pas encore été publiés.
De l'autre côté, on considère l'ensemble de difféomorphismes hamiltoniens à support compact d'une surface à bord qui préservent une configuration de cercles prédéterminée. Nous donnons des estimations de l'énergie d'Hofer d'un tel difféomorphisme qui se basent sur la complexité d'un type de tresse que nous pouvons lui attribuer. L'outil utilisé ici est la théorie d'Heegaard Floer quantitative, récemment développée par Cristofaro-Gardiner, Humilière, Mak, Seyfaddini et Smith. Les résultat dans cette direction sont déjà contenus dans un travail de l'auteur, et dans une collaboration avec Ibrahim Trifa.

Title : Braids in Low-Dimensional Hamiltonian Dynamics

Keywords : Braids, Hamiltonian systems, generating functions, Floer theory, Hofer geometry

Abstract : In this thesis we study Hamiltonian systems using the topology of their closed orbits. The results we present deal, on the one hand, with properties of the generating functions associated with a particular Hamiltonian diffeomorphism, and on the other hand, with the Hofer distance between two diffeomorphisms that realise braids of different types.
In the first context, we will rely on results by Patrice Le Calvez to show that any (up to stabilisation) generating function of a Hamiltonian diffeomorphism with compact support in the plane admits a filtration into the second tensor power of the Morse complex. By “linking filtration” we mean a filtration which associates an integer with any pair of critical points, and such when the two points are distinct the associated value is exactly the linking number of the two orbits corresponding to the critical points. It is possible to define such filtration in the context of Hamiltonian Floer theory as well, and to study its behaviour with respect to the product in homology. The author's results in this direction are still unpublished.
On the other hand, we consider the set of Hamiltonian diffeomorphisms with compact support of a surface with boundary which preserves a predetermined configuration of circles. We give estimates of the Hofer energy of such a diffeomorphism based on the complexity of a type of braid that we assign to it. The tool we use here is quantitative Heegaard-Floer theory, recently developed by Cristofaro-Gardiner, Humilière, Mak, Seyfaddini and Smith. The results in this direction are already contained in a work by the author, and in one in collaboration with Ibrahim Trifa.

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