Université Sorbonne - Paris Nord

Laboratoire Analyse, Géométrie et Applications

M2 - Mathématiques Fondamentales

Floer and Morse Homology via Generating Functions

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Introduction

The core of this work is Chapter 3: as per the title, we are going to develop relative Morse and (Lagrangian) Floer theories, and then compare them when they describe the same object. As it turns out, the two descriptions are consistent, and the isomorphism we are going to examine also respects the natural filtrations of the homology complexes.

Chapter 2, which might be read later, depending on the notions and the results the reader is already acquainted with, examines in fact under which conditions a generating function for a Lagrangian submanifold exists, and how unique it is. About the existence, a priori we do not know a lot. From Chapter 3 we already have some kind of uniqueness, since different generating functions for the same Lagrangian submanifold need to give rise to isomorphic Morse homologies (for instance, isomorphic to the same Floer homology); the uniqueness can be precisely described in terms of operations on the generating functions. Moreover, uniqueness and existence properties will remain stable under Hamiltonian isotopies of the Lagrangian submanifold, and under some further hypotheses, also under symplectic isotopies: it is the content of Sikorav's and Viterbo's Theorem.

The first chapter is just a quick survey of the necessary Differential Geometry one needs to understand the topic; it can definitely be skipped, at least until the part I dedicated to the Maslov index. This paragraph is to Floer Theory what Chapter 2 is to Morse Theory of the generating functions: the possibility of the existence of a $\mathbb{Z}$-graded Floer complex is not trivial, and needs to be justified. In the appendices we collected some tools which are necessary to approach some proofs here, and relegated there also the proof of one fundamental proposition (following Viterbo's choice in his paper [33]).

This work comes at the end of my second year in Paris, and of my fifth year at University. I do not claim originality for its content, it is mostly a rearrangement of material from previous works; the sources are reported later, in the text.

I wish to thank the FSMP, which allowed me to stay in Paris for this year offering me a grant within the project PGSM, and my advisor M. Humilière who was always available to listen to my doubts and to try to solve them together. I am looking forward to working with him as a PhD student.

Chapter 1

Symplectic Manifolds

1.1 Basic definitions in Differential Geometry

1.1.1 Manifolds, vector bundles and differential forms

In this section we are going to recall some basic definitions and constructions in Differential Geometry, omitting the proofs as it would not be the goal of the text. There is a huge number of standard references where one can find the details, for instance, in French, [16], and in English [18]. For the same reason, we are not going to provide concrete examples. $\mathbb{K}$ will always be either $\mathbb{R}$ or $\mathbb{C}$, the results will not change.

Definition 1.1.1 (Smooth manifolds)**.** A topological space X with a collection $(\mathcal{U}_j, \varphi_j)_{j\in J}$, where $\varphi_j : \mathcal{U}_j \to \mathbb{R}^{n_j}$ is a homeomorphism on its image, is a smooth manifold if it is Hausdorff, second countable and the transition maps

$$\varphi_j \circ \varphi_k^{-1} : \varphi_k(\mathcal{U}_j \cap \mathcal{U}_k) \to \varphi_j(\mathcal{U}_j \cap \mathcal{U}_k)$$

are smooth ($\mathcal{C}^\infty$), with smooth inverses. $(\mathcal{U}_j, \varphi_j)_{j\in J}$ is an **atlas** for X, and $(\mathcal{U}_j, \varphi_j)$ is a **chart**.

Of course, if X is connected, $n_j = n$ does not depend on j: in this case n is defined to be the dimension of M.

A function between smooth manifolds $f : M \to N$ is smooth if it is smooth when read through charts, i.e. all its compositions with a transition map for a fixed atlas are smooth. This notion, thanks to the chain rule for derivatives, does not depend on the specific atlas we choose, as long as they're **equivalent**: two atlases are equivalent if given charts $(\mathcal{U}, \varphi)$, (V, ψ) (one for each atlas), the transition maps are smooth (definition as above). From here on, every manifold and map will be assumed to be smooth, unless otherwise stated.

Definition 1.1.2 (Tangent space)**.** If $x \in M$, M is a manifold, the tangent space to M at x is the set of all parametric curves $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{U}_x$, where $\varepsilon > 0$

is not fixed, and $\mathcal{U}_x$ is an open neighbourhood of x which is not fixed either, quotiented by the following equivalence relation:

$$\gamma \sim \gamma' \Leftrightarrow \text{ for any chart } (U, \varphi) \text{ at x } \frac{d}{dt}_{\restriction t=0}(\varphi \circ \gamma) = \frac{d}{dt}_{\restriction t=0}(\varphi \circ \gamma')$$

The composition is always defined on a small neighbourhood of x, however small.

The tangent space at $x \in M$, denoted T_xM admits indeed the structure of a vector space with dimension $\dim T_xM = \dim M$.

Definition 1.1.3 (Differential)**.** Let $f : M \to N$ be a differentiable function (i.e. whose compositions with transition functions are differentiable), and $x \in M$. Then we can define a linear application

$$d_xf : T_xM \to T_{f(x)}N \text{ via } [\gamma] \mapsto \left[\frac{d}{dt}_{\restriction t=0} f \circ \gamma(t)\right]$$

Remark. The differential satisfies the chain rule: if $f : M \to N$, $g : N \to P$ are smooth functions, then for $x \in M$, $d_x(g \circ f) = d_{f(x)}g \circ d_xf$.

A function $f : M \to N$ is therefore an **immersion** at a point $x \in M$ if d_xf is injective; a **submersion** if d_xf has maximum rank. If f is a homeomorphism, with smooth inverse d_xf invertible at every $x \in M$, f is a **diffeomorphism**. The set of diffeomorphism between two manifolds M, N is going to be denoted Diff(M, N). Remark that by the chain rule, Diff(M):=Diff(M, M) has an obvious group structure. An immersion which is a diffeomorphism when restricted to its image is called an **embedding**.

There are several, very visual and trivial examples where a manifold contains a copy of a smaller one: see for instance circles in the plane or on a sphere. There are two possible abstractions of this notions:

Definition 1.1.4 (Embedded submanifold)**.** Let M be a manifold. An embedded submanifold (P, e) is a pair of a manifold and an embedding $e : P \to M$.

Definition 1.1.5 (Immersed submanifold)**.** Let M be a manifold. An embedded submanifold (P, i) is a pair of a manifold and an immersion $e : P \to M$.

Remark that the topology on the image of an immersion needs not be the subspace topology, if the immersion is not a homeomorphism. Remark however that if P above is compact, then i is in fact an embedding (and with more generality, proper immersions are embeddings). One can prove that immersions are local embeddings (locally on P), so that immersed submanifolds are in a way locally embedded submanifolds. Unless otherwise indicated, submanifolds will be embedded.

We are also interested in giving a notion of smoothness for functions associating to a point a tangent vector to the manifold at that point; there are several reasons to do it, at first from obvious applications from physics (they are the obvious generalisation on manifolds of objects one could work with in Euclidean spaces, such as electromagnetic fields...) and then because of theoretic interests. We give the next formal, and more general, definition:

Definition 1.1.6 (Vector bundle)**.** A smooth map $p : E \to B$ is a (smooth) vector bundle if there is an open cover of B, $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$ such that there is a diffeomorphism $\psi_\alpha : p^{-1}(\mathcal{U}_\alpha) \simeq \mathcal{U}_\alpha \times \mathbb{K}^m$ making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(\mathcal{U}_\alpha) & \xrightarrow{\psi_\alpha} & \mathcal{U}_\alpha \times \mathbb{K}^m \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathcal{U}_\alpha & \end{array}$$

We also require the following condition on the transition maps: if $\psi_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{K}^m \to \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{K}^m$, we want $\psi_{\alpha\beta}(x, v) = (x, M_{\alpha,\beta}(x)v)$, for some $M_{\alpha,\beta} \in \mathcal{C}^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta; GL_m(\mathbb{K}))$, i.e. the transition maps need to be linear on the fibres.

Remark. Remark that the transition maps satisfy the so-called **cocycle condition**: if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset$, then $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. When given an open cover $(\mathcal{U}_\alpha)_{\alpha \in A}$ of a manifold and functions $(\psi_{\alpha\beta})_{(\alpha,\beta) \in A \times A}$, $\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(\mathbb{K}^m)$ such that $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$ and that it verifies the cocycle condition, there is a vector bundle defined as a quotient $E = \coprod_{x \in M} \{x\} \times \mathbb{K}^m / \sim$, where for $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $(x, v) \sim (x, u)$ if $v = \psi_{\alpha\beta}(x)u$ (i.e. v is a vector seen through the trivialisation α, and u is the same vector but through the trivialisation β).

The fibre of a vector bundle $p : E \to B$ at $x \in B$ is $p^{-1}(x) =: E_x \simeq \mathbb{K}^m$. B is the base of the vector bundle, and if it is connected the dimension m of the fibres does not depend on the point: m is therefore called the **rank** of the vector bundle. A **section** of p is a right inverse: it will always be required to be smooth. A **local trivialisation** of the vector bundle will be the data of an open set and a diffeomorphism $(\mathcal{U}_\alpha, \psi_\alpha)$ as in the definition. A **trivialising cover** of M will be a collection of local trivialisations $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$ as in the definition above.

We need to introduce the notion of **vector bundle morphism**: if $\varphi : M \to N$ is a smooth function, $p : E \to M$, $q : F \to N$ are two smooth vector bundles, we say that a smooth function $\bar{\varphi} : E \to F$ is a morphism of bundles over φ if the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\bar{\varphi}} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{\varphi} & N \end{array}$$

and so that $\bar{\varphi}$ induces linear morphisms on the fibres: $(\bar{\varphi}_m : E_m \to F_{\varphi(n)}) \in \mathrm{Hom}_{\mathbb{K}}(E_m, E_{\varphi(m)})$. A particular class of bundle morphisms we are going to consider are those over the identity: they are simply smooth families of linear maps over the manifold, so that the base points are preserved.

The first vector bundle one can think of is the **tangent bundle**: if M is a smooth manifold, as a set, it is $TM = \coprod_{x \in M} \{x\} \times T_x M$, it can be endowed with a differential structure. The projection $p : TM \to M$ is a vector bundle, with the trivialising cover of M given by the open sets of an atlas $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$ and as local trivialisations $(\mathcal{U}_\alpha, d\varphi_\alpha)$. Smooth sections of TM will be called **vector fields** on M, and their set will be denoted as $\mathfrak{X}(M)$; alternative notations will be $\Gamma(X; TX), \Gamma(TX)$ or even $\Gamma(p)$. Given the form of these trivialisations, we can specify easily a local basis for TM: if $\mathcal{U}$ is the open set of a chart, with coordinates $(x^1, \dots, x^n)$, a basis will be denoted with $(\partial_{x^1}, \dots, \partial_{x^n})$, and for the moment they are just the derivatives at the point of the curves following a coordinate with unit speed (read through the chart with those coordinates, ∂_{x^i} corresponds to e_i in $\mathbb{R}^n$).

Given a vector bundle, one can construct others using pointwise operations, and giving the good trivialisations. Here we mention only the Whitney sum, the dualisation of a vector bundle and its k-th exterior power ($k \in \mathbb{N}$), since they will appear in the text with more importance than other bundles, and we might need to make use of the explicit trivialisations for the calculations.

If $p : E \to B$, $q : F \to B$ are vector bundles on the same base B, with trivialisations $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$, $(\mathcal{V}_\beta, \psi_\beta)_{\beta \in B}$, their **Whitney sum** $p \oplus q : E \oplus F \to B$ is the vector bundle whose fibre at $x \in B$ is $E_x \oplus F_x$. For the trivialisations we take a refinement of both the covers (for instance we can take all the possible intersections), and up to relabelling we can take it to be $(\mathcal{U}_\alpha)_{\alpha \in A}$: the local trivialisations are therefore given by $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, \varphi_\alpha \oplus \psi_\beta)_{(\alpha,\beta) \in A \times A}$.

If $p : E \to B$ is a vector bundle with trivialisation $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$, its **dual bundle** $p^* : E^* \to B$ is the vector bundle whose fibre at $x \in B$ is E_x^* and with trivialisations given by $(\mathcal{U}_\alpha, (\varphi_\alpha^{-1})^*)_{\alpha \in A}$. The local basis will be the dual of the local basis in the tangent bundle: with the same notations as above, $(dx^1, \dots, dx^n)$ are such that $dx^i . \partial_{x^j} = \delta^i_j$ (the Kronecker's symbol).

Dualising the tangent bundle of a manifold M, we find its cotangent bundle T^*M, whose importance will be massive in this text.

Another capital operation we borrow from linear algebra and apply to vector bundles is the exterior power: for a vector bundle $p : E \to B$, $\Lambda^k(E)$ is the vector bundle obtained taking pointwise the k-th exterior power, and the local trivialisations will be $(\mathcal{U}_{\alpha\beta}, \varphi_{\alpha\beta}^{\otimes k})$. Remark that the same trivialisations also define the k-th tensor power bundle, with fibre $E_x^{\oplus k}$. We shall denote it with $E^{\oplus k}$.

Sections $\lambda \in \Gamma(X; \Lambda^k(T^*M)) =: \Omega^k(M)$ are called **differential forms**, and k is their **degree**. We define $\Lambda^0(M) = \mathbb{K}$, so that $\Omega^0(M) = \mathcal{C}^\infty(M; \mathbb{K})$. They are particularly important because with the exterior derivative (we are going to define it soon) they form a complex, giving rise to a cohomology theory, whose cup-product has an easy interpretation. We start in fact from the latter, and then define the former by some basic properties.

Definition 1.1.7 (Wedge product)**.** If $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, we define $\alpha \wedge \beta(M)$ as $(\alpha \wedge \beta)_x = \alpha_x \wedge \beta_x$:

$$(\alpha \wedge \beta)_x(v_1, \dots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) \alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \beta_x(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$

where $\mathfrak{S}_{p+q}$ is the symmetric group over a set of $p+q$ elements, and $\text{sgn}(\sigma)$ is the signature of σ.

With the wedge product, $\Omega(M) = \bigoplus_{k\in\mathbb{N}} \Omega^k(M)$ forms a unital, anticommutative $\mathbb{K}$-algebra: $\wedge$ is in fact associative, anticommutative (if $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, then $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$) with neutral element $1 \in \Omega^0(M)$. Note that if the maximum degree of a nonzero differential form is the dimension of the manifold.

Remark. Since $\Lambda^\bullet(T^*M)$ is a fibre bundle, and given the form of its trivialisations, we obtain a simple local description of differential forms: if $\alpha \in \Omega^k(M)$, if $(x^1, \dots, x^n)$ are coordinates around a point $P \in M$, we can write

$$\alpha(x) = \sum_{\substack{J=\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\} \\ i_1<\dots i_k}} \alpha_{i_1,\dots,i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} =: \alpha_J(x) dx^J$$

adopting Einstein's summation convention.

We can transport differential forms in a canonical way forms via smooth functions: if $f : M \to N$ and $\omega \in \Omega^k(N)$,

$$\Omega^k(M) \ni f^*\omega : x \mapsto ((v_1, \dots, v_k) \mapsto \omega_{f(x)}(d_x f.v_1, \dots, d_x f.v_k))$$

and if $f : M \to N$, $g : N \to P$, we have $(gf)^* = f^*g^*$, i.e. one can view Ω^k as a contravariant functor from the category of smooth manifolds and functions, to the category of vector spaces. This also implies that if $\varphi \in$Diff(M, N), then φ^* is an isomorphism.

Let us define the exterior derivative:

Definition 1.1.8. There is a unique family of applications $(d_k : \Omega^k(M) \to \Omega^{k+1}(M))_{k\in\mathbb{N}}$ (which will be simply denoted as d) satisfying the following conditions:

- d is $\mathbb{K}$-linear;
- If $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta$
- $d^2 = d \circ d$;
- df coincides with the usual differential for $f \in \mathcal{C}^\infty(M; \mathbb{K})$

Clearly one should prove this assertion. We just mentions that the last condition means that $df \in \Omega^1(M)$ is defined as $x \mapsto d_x f$, and that there is a local definition in coordinates: if $(x^1, \dots, x^n)$ are coordinates on an open neighbourhood $\mathcal{U}$ of a point $P \in M$, one can check that if $\alpha \in \Omega^k(\mathcal{U})$ we have the following expression:

$$d\alpha(x^1, \dots, x^n) = \partial_{x^i} \alpha_J(x^1, \dots, x^n) dx^i \wedge dx^J$$

One can in fact check that this description verifies the four properties of the definition, and that it does define a global differential form. An alternative approach is to show the next proposition, which gives a global formula for the exterior derivative:

Proposition 1.1.1. If $\alpha \in \Omega^k(M)$, $X_0, \dots, X_k \in \mathfrak{X}(M)$, we have

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \omega(X_0, \dots, \hat{X}_i, \dots, X_k) +$$
$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

where the entries with the hats are omitted.

By the first property, $(\Omega^\bullet(M), d)$ is a complex of cochains, whose homology groups $H^\bullet_{dR}(M; \mathbb{K})$ are called **de Rham**'s cohomology groups.

We also find that d commutes with the pullback: it is a natural transformation between the functors $\Omega^\bullet$ and $\Omega^{\bullet+1}$, and if we consider the cochain complexes $(\Omega^\bullet(M), d^M)$ and $(\Omega^\bullet(N), d^N)$, and the smooth function $\varphi : M \to N$, then φ^* is a chain map, so that it defines morphisms at the homology level.

Differential forms are not the only sections we are going to make use of. The next section will be devoted to the study of vector fields, which are just sections of $TM \to M$. Closer to the spirit of differential forms however are sections of any tensor product of bundles. We are going to use, to give some definitions, the notion of a **Riemannian metric** on a manifold M: it is just a section of the vector bundle $(T^*M)^{\oplus 2} \to M$, symmetric in the arguments and which is a scalar product on each fibre. Riemannian metrics always exist (standard use of partitions of the unity, by compactness of the space of scalar products); we are not going to go any deeper in the subject.

1.1.2 Vector fields

Before proceeding with some more geometric tools, we need to do some further work on vector fields. Note that a vector field, even if locally defined (i.e. on an open set, not necessarily on the whole manifolds) defines a differential operator acting on functions: if M is a smooth manifold, $\mathcal{U}$ an open set in M, $f \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R})$, $X \in \mathfrak{X}(\mathcal{U}) = \Gamma(\mathcal{U}; TM|_\mathcal{U})$, we define

$$X \cdot f : \mathcal{U} \to \mathbb{R}, x \mapsto (X \cdot f)(x) := d_x f.X(x)$$

It is a basic result in Differential Geometry that there is a bijection between the tangent space at a point and the set of derivations at that same point. Intuitively, giving a tangent vector is the same as giving a directional derivative at that point[1], and this isomorphism can be glued properly, so that if we specify a global derivation, there is exactly one vector field associated to it. We use it to define the **Lie bracket**:

$$X, Y \in \mathfrak{X},\ [X, Y] \cdot f(x) := X \cdot (Y \cdot f)(x) - Y \cdot (X \cdot f)(x)$$

In local coordinates then $[X, Y]$ is $X^i \partial_{x^i}(Y^j)\partial_{x^i} - Y^i \partial_{x^i}(X^j)\partial_{x^j}$.

[1]Here we need to make use of the smoothness of the manifold: if it is not smooth, the vector space of derivations might be infinite-dimensional.

Given the differential interpretation of a vector field, it is immediate to see that to a vector field X corresponds the differential equation $\dot{x} = X(x)$ ($\dot{x}$ is the temporal derivative of the curve). If the vector field is smooth, we have local uniqueness and existence of the solutions by Cauchy-Lipschitz Theorem: if ϕ_X^t is the flow of the vector field X at the time t (i.e. the solution of the above differential equation), ϕ_X^t is a diffeomorphism. There are some other properties of the flow which we are going to mention later, but for now we need to make the following remarks:

- The flow at time 0 is always the identity;
- If the manifold is compact, the flow is defined for every $t \in \mathbb{R}$: we say that it is **complete**;
- If (X_t) is a smooth time-dependent family of vector fields, we still have local existence and uniqueness of the solutions, thus a well-defined flow;
- In the easiest case possible, when the manifold is compact and the vector field is **autonomous** (it does not depend on the time), the flow is a semigroup: is ϕ_X^t, ϕ_X^s are as above, by uniqueness we have

$$\phi_X^t \circ \phi_X^s = \phi_X^{t+s} = \phi_X^s \circ \phi_X^t, \;\; (\phi_X^t)^{-1} = \phi_X^{-t}$$

As for differential forms, we can transport between manifolds the vector fields, but in this case only through diffeomorphisms: if M, N are smooth manifolds, $\varphi : M \to N$ is a diffeomorphism, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, then:

$$\varphi^* Y \in \mathfrak{X}(M), \;\; \varphi^* Y(p) = d_{\varphi(p)}\varphi^{-1}.Y(\varphi(p)),$$
$$\varphi_* X \in \mathfrak{X}(N), \;\; \varphi_* X(q) = d_{\varphi^{-1}(q)}\varphi.X(\varphi^{-1}(q))$$

Remark that $\varphi_* = (\varphi^{-1})^*$.

We can define the Lie derivative for vector fields: if $X; Y \in \mathfrak{X}(M)$,

$$\mathcal{L}_X Y = \frac{d}{dt}_{\restriction t=0} (\phi_X^{-t})_* Y \in \mathfrak{X}(M)$$

With some work, one can find the basic identity $\mathcal{L}_X Y = [X, Y]$, which shows for instance that $\mathcal{L}_X Y = -\mathcal{L}_Y X$, which was far from being evident.

Let us look again at the definition of the action of a vector field on a function: one could see it as a map defined on coboundaries of degree 1, returning a smooth function. We can generalise the process, defining the **interior product**: if $X \in \mathfrak{X}$, $\iota_X : \Omega^\bullet(M) \to \Omega^{\bullet -1}(M)$,

$$\omega \in \Omega^k(M), \;\; \iota_X \omega(v_1, \ldots, v_{k-1}) = \omega(X, v_1, \ldots, v_{k-1})$$

We mention only two key properties of the interior product: if $X, Y \in \mathfrak{X}(M)$, $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$,

$$\iota_X \iota_Y = -\iota_Y \iota_X, \;\; \iota_X(\alpha \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^k \alpha \wedge \iota_X \beta$$

We conclude by talking about the Lie derivative for differential forms: if $\omega \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$, the Lie derivative $\mathcal{L}_X\omega$ is defined formally almost the same way as the one for vector fields (they coincide if the vector field is autonomous):

$$\mathcal{L}_X\omega = \frac{d}{dt}_{\restriction t=0}(\phi_X^t)^*\omega \in \Omega^k(M)$$

The two most important properties are the following: $\mathcal{L}_X$ commutes with d and the pullbacks (one could say it is natural chain map), and above all **Cartan's Magic Formula**: $\mathcal{L}_X = d\iota_X + \iota_X d$. To prove it, one can check that the definition and the formula above are both derivations of degree 1 commuting with d and distributing over wedge products, which agree on functions and differential of functions: given the local expression of differential forms, this proves the identity on the whole $\Omega(M)$.

1.1.3 Distributions and foliations

Distributions and foliations can be seen as two sides of the same coin: the former as the differential, local side, and the latter as the integral and local side; however, not all distributions can be integrated (with the meaning we are going to discuss later), so the relations between these objects are going to be much more subtle than this. We refer to [18] for a thorough explanation of these concepts.

Definition 1.1.9 (Subbundle)**.** If $p : E \to B$ is a vector bundle, $q : D \to B$ is a subbundle of p if q is a vector bundle there is a bundle morphism over the identity, $\Phi : D \to E$ which is injective on the fibres.

A subbundle is basically a smooth (on M) assignation of vector subspaces of the fibres of E.

Definition 1.1.10 (Distribution)**.** A distribution of rank k on a manifold M is a smooth subbundle of TM of rank k.

They can be locally described by a set of (locally defined) vector fields: if D is a distribution on M of rank k, $x \in M$, there are k locally defined independent vector fields $X_1, \dots, X_k$ such that $D0\,\mathrm{Span}(X_1, \dots, X_k)$. The dual approach is also valid: if D is a distribution of rank k on a manifold M^n, $x \in M$, there are locally defined 1-forms $\lambda_1, \dots, \lambda_{n-k}$ around x such that $D_x = \bigcap_{i=1}^{n-k} \ker \lambda_i$.

The first distribution that comes to mind is then, given a vector field, the assignation to every point of a manifold of the vector space spanned by the vector field at that point: this is indeed a distribution of rank 1. Another idea then comes to mind: a vector field is integrable: if $x \in M$, $X \in \mathfrak{X}(M)$, as we saw there is an integral curve, locally defined through X. An integral curve is in general an immersed submanifold: there are examples where they are in fact not embedded (think of dense trajectories on a torus). We want to generalise to higher dimensions:

Definition 1.1.11 (Integrable distribution)**.** A distribution D on M is integrable if for every point of $x \in M$ there is an immersed submanifold $i_x : P_x \to M$ such that $x \in i(P_x)$, $T_x i(P_x) = D_x$. (P_x, i_x) is an **integral manifold** for D.

Simple examples show that not all distributions are integrable, so we wold like to find a local criterion. Remark that if a distribution D on M is integrable, $x \in M$ and X, Y are local vector fields defined around x such that $X, Y \in D$, then $[X, Y] \in D$, since tangent spaces to immersed submanifolds are closed under Lie bracket. Such condition is called **involutivity** of a distribution. Turns out, the involutivity is equivalent to the integrability of D:

Theorem 1.1.2 (Frobenius)**.** An involutive distribution is integrable.

Proof. Omitted. □

Actually, Frobenius's Theorem goes a bit beyond that: a chart $(\mathcal{U}, \varphi = (x^1, \ldots, x^n))$ is flat for the distribution D of rank k if $\varphi(\mathcal{U})$ is a cube in $\mathbb{R}^n$, and if D, read in coordinates (x), is spanned by the first k vectors of the canonical basis. This implies that we have an explicit local description of the local integral submanifolds: they are k dimensional spaces satisfying the equations $x^i = c^i$ for $i = k+1, \ldots, n$, and a set of constants c^i. A distribution D is **completely integrable** if there is an atlas of flat charts for D. Frobenius's Theorem in fact shows that a involutive distribution is completely integrable. The following proposition links distributions with foliations:

Proposition 1.1.3. If D is a distribution of rank k on M^n, and $(\mathcal{U}, x^1, \ldots, x^n)$ a flat chart for D, H an integral manifold of D, then $H \cap \mathcal{U}$ in the coordinates (x^i) is a countable disjoint union of parallel spaces of dimension k, each of which is open in H and embedded in M.

In fact, this is exactly the phenomenon we see with the dense orbits on the torus: it is a tautology if we use the flat torus description $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and the picture shows is for the torus embedded in $\mathbb{R}^3$.

Consider now any collection $\mathfrak{F}$ of k-dimensional embedded submanifolds of M^n. A chart $(\mathcal{U}, \varphi = (x^1, \ldots, x^n))$ around a point $p \in M$ is flat for $\mathfrak{F}$ if $\varphi(\mathcal{U})$ is a cube of $\mathbb{R}^n$ and each submanifold in $\mathfrak{F}$ intersects $\mathcal{U}$ in either the empty set or in a countable union of k-dimensional spaces satisfying the equations $x^i = c^i$ for $i = k+1, \ldots, n$, and a set of constants c^i.

Definition 1.1.12 (Foliation)**.** If M^n is a manifold, a partition of M $\mathfrak{F}$ of disjoint, connected and k-dimensional immersed submanifolds is a foliation if there is an atlas of M of flat charts for $\mathfrak{F}$. The elements of $\mathfrak{F}$ are called **leaves** of the foliation.

The trivial example of foliation is the subdivision of $\mathbb{R}^n$ in parallel k-planes, but one can also think of a partition in concentric spheres of $\mathbb{R}^n$, or again of the torus: if we adopt the flat point of view, one has foliations given by the images of straight lines under the quotient, whether the angular coefficient is rational or not is of no importance; it suffices to remark that if it is then the submanifolds are embedded (images of a circle on the torus, in fact), and that if it is not then they are simply immersed.

It is clear that if $\mathfrak{F}$ is a foliation on M, the tangent spaces to its leaves form a completely integrable distribution. The converse is also true:

Theorem 1.1.4 (Frobenius for foliations)**.** If D is an involutive distribution on M, the collection of its maximal integral manifolds is a foliation on M.

1.1.4 Transversality

For a quick introduction to transversality, we refer to [17] (with some applications to Morse Theory), to [18] for a purely geometrical approach and to [15] for applications to spaces of (not necessarily smooth) functions.

Definition 1.1.13. Let M, N be smooth manifolds, $f \in \mathcal{C}^\infty(M;N)$, P a submanifold of N. Then we say that f is transverse to P, and write $f \pitchfork P$, if $\forall x \in f^{-1}(P)$ the following holds:

$$d_x f(T_x M) + T_{f(x)} P = T_{f(x)} N$$

For a set $S \subseteq M$ and a submanifold P of N, we say that f is transverse to P along S if the above relation holds true for any $x \in f^{-1}(P) \cap S$.

If P, Q are two submanifolds of N, if $e : Q \to N$ is an embedding, we say that Q and P are transverse (w.r.t. e) if $e \pitchfork P$, which in terms of tangent spaces is translated to

$$T_x Q + T_x P = T_x N \quad \forall x \in Q \cap P$$

and we shall write $Q \pitchfork P$ (omitting the embedding from the notation).

A basic property of transversality is:

Lemma 1.1.5. Let M, N be two smooth manifolds, $f \in \mathcal{C}^\infty(M;N)$, P a submanifold of N. Then if $f \pitchfork P$, the following hold:

i) $f^{-1}(P)$ is a submanifold of M;

ii) the differential of f at any point $x \in f^{-1}(P)$ induces an isomorphism between the normal spaces:

$$d_x f : T_x M / T_x f^{-1}(P) \xrightarrow{\sim} T_{f(x)} N / T_{f(x)} P$$

Given the lemma, one can say a bit more about the dimensions in the first condition using the second one: if $f \pitchfork P$ (same setting as in the lemma), then the codimension of $f^{-1}(P)$ in M equals the codimension of P in N (from the isomorphism in the second part of the lemma).

1.2 Some notions of Symplectic Geometry

Here we are going to give more detail, but still the aim of this text is not to provide a comprehensive introduction to the geometry of symplectic spaces or manifolds; the reader can refer, for instance, to [22] or [9].

1.2.1 Symplectic linear Algebra

Definition 1.2.1 (Symplectic vector space)**.** If V is a real vector space endowed with a bilinear form ω, it is symplectic (and ω is said to be symplectic too) if ω is antisymmetric and non-degenerate.

The prototype of symplectic vector space, and then of symplectic manifold, is $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ (as $\mathbb{R}$-vector spaces clearly) endowed with the standard symplectic form ω_{st}: if $(x^i, \sqrt{-1}x^i)_{i=1}^n$ is a basis for $\mathbb{C}^n$, we define $\omega_{st}(v,w) = \sum_{i,j=1}^n v^i w^j - v^j w^i$ where the index i refers to the coordinate ∂_{x^i}, and j to $\partial_{\sqrt{-1}x^j}$. Denoting $y^j = \sqrt{-1}x^j$, $dz^j = dx^j + \sqrt{-1}dy^j$, $d\bar{z}^j = \overline{dz^j}$, we can write:

$$\omega_{st} = \sum_{j=1}^n dx^j \wedge dy^j = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz^j \wedge d\bar{z}^j$$

If one writes $(\cdot,\cdot)$ for the standard $\mathbb{R}$-bilinear scalar product on $\mathbb{C}^n$, by non-degeneracy of both ω_{st} and $(\cdot,\cdot)$, one immediately sees that there needs to be a linear application $J_{st} : \mathbb{C}^n \to \mathbb{C}^n$ verifying $\omega_{st} = (J_{st}\cdot,\cdot)$; with some easy calculation one sees that J_{st} is simply the multiplication by $\sqrt{-1}$, and that it has the matrix, in these coordinates,

$$J_{st} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

$\mathbf{0}$ and $\mathbf{1}$ are respectively the 0 and identity matrix of dimension n. J_{st} will be called **standard almost complex structure**. Another feature it has is that $J_{st}^*\omega_{st} = \omega_{st}(J_{st}\cdot, J_{st}\cdot) = \omega$. More generally:

Definition 1.2.2 (Almost complex structure)**.** Let (V,ω) be a symplectic vector space, $J \in \mathrm{End}(V)$ is an almost complex structure if $J^2 = -Id$. Moreover, it is calibrated by ω if it is **symplectic**, i.e. $J^*\omega = \omega$ and $\omega(\cdot, J\cdot)$ is a scalar product.

Clearly, J_{st} is calibrated by ω_{st}. Of course, given a symplectic vector space (V,ω), a priori there might be no almost complex structure calibrated by ω: however, one can prove the following result, with the same idea we used in the construction of J_{st}:

Lemma 1.2.1. If (V,ω) is a symplectic vector space, there is a continuous surjective map from the set of scalar products on V to that of almost complex structures calibrated by ω, $\mathcal{J}_c(\omega)$, and $\mathcal{J}_c(\omega)$ is contractible.

We finish this paragraph recalling some standard vocabulary we need when dealing with bilinear form, and more so in the symplectic case: if (V,ω) is a symplectic form, $W \leq V$ a subspace, then:

- $W^\perp := \{\, v \in V \mid \forall w \in W,\ \omega(w,v) = 0 \,\}$, it is another subspace of V;
- if $W \leq W^\perp$, W is **isotropic** (and $\omega|_W = 0$);
- if $W^\perp \leq W$, W is **coisotropic**;

- if $W \cap W^{\perp} = 0$, W is **symplectic** (as the restriction of ω is still non-degenerate);
- if W is a maximal isotropic subspace, of minimal coisotropic subspace, W is **lagrangian**: if and only if it verifies $W = W^{\perp}$.

Of course, if $N = W \cap W^{\perp}$, then W/N and $W^{\perp}/N$ are always symplectic.

The Symplectic Group

We are interested in the isometries of a symplectic vector space: we define

$$Sp(2n) = Sp(2n, \mathbb{R}) := \left\{ \varphi \in \mathrm{End}(\mathbb{R}^{2n}) \;\middle|\; \varphi^* \omega_{st} = \omega_{st} \right\}$$

One can give a similar definition for $Sp(V, \omega)$, but it is not as interesting thanks to the following lemma:

Lemma 1.2.2 (Normal form)**.** If (V, ω) is symplectic, then there is a **symplectic basis** of V $(v_i, w_j)_{i,j=1}^n$: $\omega(v_i, w_j) = \delta_{i,j}$. Moreover, if $W \leq V$, there is a symplectic basis of V, (v_i, w_j), and integers $k, l \in \mathbb{N}$ such that

$$W^{\perp} = \mathrm{Span}(v_1, \ldots, v_{k+l}, w_1, \ldots, w_l)$$
$$W^{\perp} = \mathrm{Span}(v_{k+1}, \ldots, v_n, w_{k+l+1}, \ldots, w_n)$$
$$W \cap W^{\perp} = \mathrm{Span}(v_{k+1}, \ldots, v_{k+l})$$

Proof. Omitted. □

The first part of this Lemma says that there is a linear isomorphism $\varphi : \mathbb{R}^{2n} \to V$ such that $\varphi^* \omega = \omega_{st}$, so that all the symplectic vector space are isometrically isomorphic, and their symplectic groups therefore isomorphic.

Remark. A consequence is that a symplectic vector space is always even dimensional. One could still see it with a more abstract argument: if J is an almost complex structure on V, then we must have $\det(J)^2 = \det(-Id_V) = (-1)^{\dim(V)}$, so that $\dim(V)$ needs to be even.

We can give an injection of $GL_(n, \mathbb{C})$ into $GL(2n, \mathbb{R})$: via the basis we chose, a complex number $v + iw$ is identified with the vector ${}^t(v\ w)$, so that multiplication by i has, as we said, matrix J_{st} and that a matrix $X + iY \in GL(n, \mathbb{C})$ is written under this isomorphism as

$$X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

A sanity check shows that elements in the image of $GL(n, \mathbb{C})$ do commute with J. What we can show then is that:

Lemma 1.2.3. *$Sp(2n) \cap O(2n) = O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(2n) = \mathcal{U}(n)$. $O(2n)$ is the orthogonal group for $\mathbb{R}^{2n}$, and $\mathcal{U}(n)$ the unitary group for $\mathbb{C}^n$.*

Proof. It is just a sequence of calculations using basic definitions of the groups. □

This Lemma finds its usefulness in the proof of a continuous and unique decomposition of a symplectic matrix A in a product $A = PU$, with P symmetric, symplectic and positive definite, $U \in \mathcal{U}(n)$. Given the existence of roots for symmetric and positive definite matrices, we find a retraction of $Sp(2n)$ onto $\mathcal{U}(n)$: the inclusion $\mathcal{U}(n) \hookrightarrow Sp(2n)$ gives a homotopy equivalence[2], so that $Sp(2n)$ is connected ($\mathcal{U}(n)$ is) and $\pi_1(\mathcal{U}(n)) \simeq \pi_1(Sp(2n))$. It is a standard fact that $\det : \mathcal{U}(n) \to \mathbb{S}^1$ induces an isomorphism of the fundamental groups: $\pi_1(Sp(n)) \simeq \mathbb{Z}$. This will be crucial in the definition of the Maslov index.

1.2.2 Symplectic manifolds

A symplectic manifold should clearly be something that locally resembles a symplectic vector space: the next definition formalises this concept properly.

Definition 1.2.3 (Symplectic manifold)**.** A symplectic manifold is a pair (M, ω), where M is a smooth manifold and ω is a smooth 2-form, which is non-degenerate and closed. ω is then a symplectic form on M.

By the remarks we have done above, since TM is locally trivial, there always is a local almost complex structure on TM, so that M needs to have even dimension as in the linear case.

To find a motivation for the fact that a symplectic form is closed, one can look for instance at [2], or at Henry Cohn's webpage[3]: the former remarks that ω_{st} is exact, in the same coordinates as above

$$\omega_{st} = -d\lambda_{st}, \;\; \lambda_{st}(x, y) = \sum_{i=1}^{n} y^i dx^i$$

but that there cannot be exact and non-degenerate 2-forms on compact manifolds, so that one needs to be satisfied with local exactness, i.e. closedness. The latter brings instead physical arguments to require the closedness.

A diffeomorphism $\varphi \in \mathrm{Diff}M$, if (M, ω) is symplectic, is said to be a symplectic diffeomorphism, or **symplectomorphism**, if $\varphi^*\omega = \omega$. The group of symplectomorphisms will be denoted $\mathrm{Sympl}(M, \omega)$ (by the properties of the pullback, $\mathrm{Sympl}(M, \omega)$ is closed under composition and inverse).

The clear generalisation of an almost complex structure is then that of a bundle morphism over the identity $J : TM \to TM$ such that $J^2 = -Id$. The notion of an almost complex structure calibrated by the form ω is then generalised the same way: a bundle morphism over the identity $J : TM \to TM$ is an almost complex structure calibrated by ω if $J^2 = -Id$, $J \in \mathrm{Sympl}(M, \omega)$

[2]We never really talked about the topology on these groups. They are subgroups of $M_{2n}(\mathbb{R}) \simeq \mathbb{R}^{4n^2}$, and they can be equipped with the induced topology. In fact, one can do better, and give them a structure of Lie groups.

[3]`https://math.mit.edu/~cohn/Thoughts/symplectic.html`

and $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. Since in point by point $x \in M$ we know that $\mathcal{J}_c(\omega_x)$ is contractible, there always exists a calibrated almost complex structure on a symplectic manifold.

Since ω is non-degenerate, if $H \in \mathcal{C}^\infty(M;\mathbb{R})$, one can define the symplectic gradient of H, also called **Hamiltonian vector field** (and H will be a **Hamiltonian**, in this case autonomous since it does not depend on the time), via the equality

$$dH = \iota_{X_H}\omega$$

X_H is the symplectic gradient. The definition is exactly the same for time-dependent Hamiltonians, that is for smooth families of smooth functions $(H_t)_{t\in J}$, J interval of $\mathbb{R}$ (not necessarily compact). In this text we are mostly going to consider periodic Hamiltonian, namely families $(H_t)_{t\in I}$ (I in this text will be the unit interval in $\mathbb{R}$, $[0,1]$) such that $H_0 = H_1$; alternatively, we can consider the smooth function $H : \mathbb{S}^1 \times M \to \mathbb{R}$, $(t,x) \mapsto H_t(x)$, and the symplectic gradient will be obviously taken on the second term.

Given a Hamiltonian vector field, one can define its flow as in the general case: for a (not necessarily autonomous) Hamiltonian H, using the identity

$$\frac{d}{ds}_{\restriction s=t}(\phi^s_{X_H})^*\omega = (\phi^t_{X_H})^*\mathcal{L}_{X_H}\omega \tag{1.1}$$

holding actually for arbitrary differential forms, one verifies that the flow of a Hamiltonian vector field is symplectic. Denote then

$$\mathrm{Ham}(M,\omega) := \left\{ \varphi \in \mathrm{Diff}(M) \;\middle|\; \exists H \in \mathcal{C}^\infty(I\times M;\mathbb{R}) : \varphi = \phi^1_{X_H} \right\}$$

We just showed the inclusion $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Sympl}(M,\omega)$. With some algebraic identities on the flows of general vector fields, one can prove without difficulty that $\mathrm{Ham}(M,\omega) \trianglelefteq \mathrm{Sympl}(M,\omega)$, but the equality is in general not true.

Darboux's Theorem and Moser's Trick

We are going to state Darboux's Theorem, omitting almost entirely the proof: we are going however to present the so-called Moser's Trick, as it will appear again in the text.

Theorem 1.2.4 (Darboux)**.** Let M be a smooth manifold, $W \hookrightarrow M$ a compact submanifold, $\omega_0, \omega_1 \in \Omega^2(M)$ symplectic forms whose restrictions to W coincide. Then there are two neighbourhood of W in M, $\mathcal{U}_0$ and $\mathcal{U}_1$, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi|_W = Id_W$, and $\varphi^*\omega_1 = \omega_0$.

Corollary 1.2.5. If (M^{2n},ω) is a symplectic manifold, for all $x \in M$ there is an open neighbourhood $x \in \mathcal{U}$ and a diffeomorphism $\varphi : \mathcal{U} \to \mathcal{V} \subseteq \mathbb{R}^{2n}$ such that $\varphi^*\omega_{st} = \omega$.

Proof. x is a compact submanifold of M. Choosing a local chart $\psi : \mathcal{U}'_0 \to V'_0 \subseteq \mathbb{R}^{2n}$ of M at x, and applying the Normal Form theorem, there is a linear isomorphism $A \in GL(2n,\mathbb{R})$ such that $A^*\omega_{st} = (\psi^{-1})^*\omega_x$. By Darboux's

Theorem there are two neighbourhoods of x, $\mathcal{U}_0$ and $\mathcal{U}_1$, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*(\psi^{-1})^*\omega = A^*\omega_{st}$. We can then choose the local chart $(\mathcal{U}_0' \cap \psi^{-1}(\mathcal{U}_0), \bar{\psi})$, with $\bar{\psi} = A\varphi\psi$. □

Moser's Trick plays a fundamental role in the proof of the Theorem. What one does, in fact, is interpolate two symplectic forms, defining the diffeomorphism as the flow of a vector field built by hand, using the non-degeneracy of the forms. More explicitly, suppose we have two symplectic forms ω_0 and ω_1 on the manifold, and that we want to find a $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*\omega_1 = \omega_0$. We can consider the path of symplectic forms $\omega_t = t\omega_1 + (1-t)\omega_0$ (there is a more general case, where we consider any path provided that the cohomology class is constant, but then one needs to use some elliptic theory to make sure that everything is smooth). To find φ it suffices to find a path of diffeomorphisms φ_t such that $\varphi_t^*\omega_t = \omega_0$; deriving both sides over time and applying the identity 1.1, one is left with the goal of solving the equation

$$\dot{\omega}_t + \mathcal{L}_{X_t}\omega_t = 0$$

where $X_t(x) = \frac{d}{ds}_{\restriction s=t}\varphi_s(x)$ is the vector field we want. Using Cartan's Magic Formula and closedness of ω we find

$$\dot{\omega}_t + d\iota_{X_t}\omega_t = 0$$

Since $\dot{\omega}_t = \omega_1 - \omega_0$ is exact (we are not going to justify this, this fact relies on a lemma by Poincaré), let $\alpha \in \Omega^1(M)$ such that $d\alpha = \dot{\omega}_t$. Then it suffices to solve the equation

$$\alpha + \iota_{X_t}\omega_t = 0$$

Some concrete examples

Remark that, whereas every manifold admits one (and therefore infinitely many) Riemannian metric, there are definitely many that do not admit a symplectic form: as we said, for instance every odd-dimensional manifold cannot be symplectic. We are going to give now two examples of manifolds which admit, naturally, a symplectic structure. The fact that $\mathbb{C}^n$ with its standard symplectic structure is a symplectic manifold is totally evident.

The cotangent bundle Let M be a smooth manifold: T^*M carries a **tautological form** λ, defined as follows: if $q \in M$ and $p \in T_x^*M$,

$$\lambda_{(q,p)}(v) = \langle p, d_{(q,p)\pi.v}\rangle$$

where $\pi : T^*M \to M$ is the natural projection. In local coordinated (q^i, p^j), a calculation shows that

$$\lambda_(q,p) = \sum_{i=1}^{n} p^i dq^i$$

λ is also sometimes called Liouville 1-form. Remark that it resembles closely the primitive of the standard symplectic form on $\mathbb{R}^{2n}$, λ_{st}. If we then define $\omega = -d\lambda$, we obtain an exact 2-form which is non-degenerate, since its coordinate expression coincides with ω_{st}. Notice that the Darboux's neighbourhoods are then simply open sets of charts, and that now we can view $\mathbb{C}^n$ as $T^*\mathbb{R}^{2n}$: the definitions of ω and ω_{st} coincide.

The tautological form is also natural: if $\varphi \in \mathrm{Diff}(M)$, it lifts to a symplectic diffeomorphism $\tilde{\varphi} \in \mathrm{Sympl}(T^*M, \omega)$, as $\tilde{\varphi} = (\varphi, (d\varphi^{-1})^*)$. $\tilde{\varphi}$ actually already preserves the 1-form λ: $\tilde{\varphi}^*\lambda = \lambda$.

Complex projective space We define $\mathbb{C}P^n$ to be $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$, that is the set of complex subspaces of dimension 1 in $\mathbb{C}^{n+1}$, with the quotient topology. Normalising the vectors, it can be shown to be homeomorphic to the quotient $\mathbb{S}^{2n+1}/\mathbb{S}^1$.[4] It is then easy to see that $\mathbb{C}P^n$ is Hausdorff and compact, we need to find local charts. Let us write $z^j = x^j + \sqrt{-1}y^j$. To give an atlas for $\mathbb{C}P^n$, consider the open sets

$$U_i = \left\{ [z^0 : \cdots : z^n] \in \mathbb{C}P^n \;\middle|\; z^i \neq 0, (z^j)_{j=0}^n \in \mathbb{S}^{2n+1} \right\}, \;\; i = 0, \ldots, n$$

Fix i and write, for $[z^0 : \cdots : z^n] \in U_i$, $x^j + \sqrt{-1}y^j = \frac{z^j}{z^i}$. We have the chart

$$\varphi_j : U_j \to \mathbb{R}^{2n}, \; [z^0 : \cdots : z^n] \mapsto \left(x^0, y^0, \ldots, y^{j-1}, x^{j+1}, \ldots, y^n\right)$$

One can check that the transition maps are indeed smooth, and even analytic.

$\mathbb{C}P^n$ is endowed with a symplectic structure: we can for instance see $\mathbb{C}P^n$ as a Kähler manifold, so that ω is the characteristic form associated to the Kähler metric (which is called Fubini-Study metric), or otherwise see $\mathbb{C}P^n$ as the symplectic reduction of $\mathbb{S}^{2n+1}$ via the symplectic and free action of the Lie group $\mathbb{S}^1$ on $\mathbb{C}^{n+1}\setminus\{0\}$. The latter point of view gives us an easy way to compute $\omega = \omega_{FS}$ (FS stands for Fubini-Study): if $p : \mathbb{C}^{n-1}\setminus\{0\} \to \mathbb{C}P^n$, $v, w \in T_{[q]}\mathbb{C}P^n$, we take a point $x \in \mathbb{S}^{2n+1}$ wuch that $p(x) = [q]$, vectors $v', w' \in T_x\mathbb{C}^{n+1}$ such that $d_xp.c' = v$, $d_xp.w' = w$, and set $\omega_{[q]}(v, w) = \omega_{st,x}(v', w')$. One can check this is a good definition; moreover, the two definitions we gave for ω_{FS} (via Kähler manifolds and symplectic reduction) differ by a factor of π.

Lagrangian submanifolds

We extend the definitions given in the section 1.2.1 for the different kind of subspaces of a symplectic space to submanifolds: a submanifold is isotropic if its tangent space is isotropic (for the symplectic form restricted on the submanifold), and so on. In particular, a submanifold is lagrangian if its tangent space is lagrangian: we remark that this condition implies that a lagrangian submanifold has always the dimension which is a half of the dimension of the symplectic manifold (it is a well known fact in linear algebra that maximal isotropic subspaces

[4] It is actually better than that: we have a Serre fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{C}P^n$ we can for instance use to compute the homotopy groups of $\mathbb{C}P^n$.

for a non degenerate antisymmetric bilinear form satisfy the same condition on the dimensions). Our exposition will be focused on lagrangian submanifolds, and in particular will be about their rigidity under hamiltonian, and lagrangian, isotopies (everything will be duly defined later on).

Symplectic reduction

Let $W \leq (\mathbb{C}^n, \omega_{st})$ be a coisotropic subspace. Then as we already noticed, there is a canonical symplectic structure $\bar{\omega}$ on the quotient $W/W^\perp$: we say that $(W/W^\perp, \bar{\omega})$ is the **symplectic reduction** of $(\mathbb{C}^n, \omega_{st})$ with respect to W. Now, if L is a lagrangian subspace of $\mathbb{C}^n$, then the $(L \cap W + W^\perp)/W^\perp$ is also lagrangian: in fact it is lagrangian before the quotient, and it is easy to see that the equality of the space with its orthogonal still holds after the projection.

Now, let W be a coisotropic submanifold in a symplectic manifold (M, ω). This submanifold gives rise to a isotropic distribution on W, $(TW)^\perp$, of rank $\operatorname{codim} W$.

Theorem 1.2.6. The distribution $(TW)^\perp$ is integrable.

Proof. Let $p \in W$, $X, Y \in (TW)^\perp$ be locally defined vector fields around p. If $v \in T_pW$, we consider a local vector field Z such that $Z(p) = v$. We know that ω is closed: we apply Proposition 1.1.1 and obtain:

$$0 = d\omega(X, Y, Z) = \omega([X, Y], Z)$$

and an evaluation at p concludes. The other terms in the expression disappear since W is coisotropic and by definition of the orthogonal. □

There is then a natural foliation for W. Identifying the leaves does not, in general, lead to a smooth structure, let alone to a symplectic manifold! However, if we make the assumption that at every point p of W there is a submanifold S which contains p and intersects every leaf exactly once, in a way that $T_pS \oplus T_pW^\perp = T_pW$, and if furthermore the quotient $\overline{W} = W/\sim$ obtained identifying all the points, on each leaf, is Hausdorff, then:

Proposition 1.2.7. Under this assumptions on W, there is a unique symplectic structure on $\overline{W}$, $\overline{\omega}$, such that if $\pi : W \to \overline{W}$ is the quotient projection and $e : W \to M$ is the embedding of W into M, then $\pi^*\overline{\omega} = e^*\omega$.

Proof. Omitted, see [22]. □

If W satisfies these conditions, we say that W is **regular**. Let now L be a lagrangian submanifold of M: we say that L and W **intersect cleanly** if $W \cap L$ is a submanifold of W and if, for every point $p \in W \cap L$, $T_pW \cap T_pL = T_p(W \cap L)$. In particular, if $L \pitchfork W$, they intersect cleanly. Let $\overline{L}$ be the set of leaves of $(TW)^\perp$ which intersect L quotiented by the equivalence relation above: then

Proposition 1.2.8. If W is regular and L and W intersect cleanly, then $\overline{L}$ is an immersed lagrangian submanifold of $\overline{W}$. If the intersection is transverse, it the lagrangian immersion is $L \cap W \to \overline{W}$. If the intersections of L with the leaves are connected, then $\overline{L}$ is an embedded lagrangian submanifold.

Proof. Always in [22]. □

$\overline{W}$ is called the **symplectic reduction** of (M, ω) with respect to the submanifold W.

1.2.3 The Maslov index

In $\mathbb{R}^{2n}$

To grade the Floer complex we will need to define the so called Maslov index. We shall follow the exposition in [22].

We can define a Maslov index for paths in $Sp(2n)$ as follows: we know that $Sp(2n)$ deformation retracts on $U(n)$ via PQ factorisation (we write a symplectic matrix a product of a symmetric, positive definite matrix and a symplectic, orthogonal matrix, we can then shrink the symmetric matrix to the identity), and that $\pi_1(U(n)) \simeq \mathbb{Z}$. In particular, if $\Psi \in Sp(2n)$, we have the explicit formula for its image under the retraction, which we shall call r: $r(\Psi) = (\Psi\, {}^t\Psi)^{-\frac{1}{2}}\Psi$. Since $Sp(2n) \cap O(2n) = U(n)$ and that $U(n)$ embeds into $GL(n, \mathbb{R})$ via

$$X + iY \mapsto \tau(X + iY) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

we can define a map $\rho : Sp(2n) \to \mathbb{S}^1$ by $\rho = \det \circ \tau^{-1} \circ r$.

Definition 1.2.4 (Maslov index ($Sp(2n)$))**.** Given a loop $\gamma : \mathbb{S}^1 \to Sp(2n)$, we define its Maslov index as $\mu(\gamma) = \deg[\rho \circ \gamma]$, where deg is the degree of the application.

Theorem 1.2.9. The Maslov index has the following properties:

Homotopy: Two loops in $Sp(2n)$ have the same Maslov index iff they're homotopic.

Product: If γ, η are two loops in $Sp(2n)$, then $\mu(\gamma * \eta) = \mu(\gamma) + \mu(\eta)$ (in particular the Maslov index of the constant loop is 0).

Direct sum: If $n = n_1 + n_2$, we can identify $Sp(2n_1)$ and $Sp(2n_2)$ as subgroups of $Sp(2n)$. Then if Ψ_j is a path in $Sp(2n_j)$, we have $\mu(\Psi_1 \oplus \Psi_2) = \mu(\Psi_1) + \mu(\Psi_2)$.

Normalisation: If $\gamma(t) = \exp(2\pi i t)$ is a path in $U(1) \subset Sp(2)$, then $\mu(\gamma) = 1$.

Proof. The first two properties are easy implications of the fact that ρ induces an isomorphism between fundamental groups (as a composition of applications inducing isomorphisms). The other two properties are obvious. □

One can also prove that the Maslov index is uniquely determined by these properties. The uniqueness gives us an interesting alternative definition of the Maslov index: for a decomposition of a symplectic matrix in four blocks

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we consider the set $\overline{Sp}_1(2n)$ of the symplectic matrices such that $\det(B) = 0$. It is a hypersurface in $Sp(2n)$, and for a path Ψ in $Sp(2n)$ we say that t is a crossing time of Ψ if $\Psi(t) \in \overline{Sp}_1(2n)$. For t, a crossing time of Ψ, we define the crossing form $\Gamma(\Psi, t) : \ker B(t) \to \mathbb{R}$ as

$$y \mapsto -\langle \dot{B}(t)y | D(t)y \rangle$$

A crossing is said to be regular when its crossing form is non-degenerate, and if a crossing (Ψ, t) is regular we define its signature $\mathrm{sign}\Gamma(\Psi, t)$ the usual way. We define the Maslov index of a path Ψ with the regular crossing times $t_1, \dots, t_k$ as $\mu(\Psi) = \frac{1}{2}\sum_{i=1}^{k} \mathrm{sign}\Gamma(\Psi, t_i)$. One can prove that μ depends only on the homotopy class of Ψ (hence it is defined for any loop) and that it satisfies to the axioms of Theorem 1.2.9, and therefore the two definitions we gave coincide.

We are also interested in giving a Maslov index to loops in the lagrangian Grassmanian, the set of lagrangian subspaces[5], of $\mathbb{R}^{2n}$ with the standard symplectic structure, denoted $Gr(Lag(\mathbb{R}^{2n}, \omega))$. We first need to describe properly $Gr(Lag(\mathbb{R}^{2n}, \omega))$.

Lemma 1.2.10. Let $L \leq \mathbb{R}^{2n}$, and assume that $L = \operatorname{Im} Z$, for

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in M_{2n,n}(\mathbb{R})$$

Then $L \in Gr(Lag(\mathbb{R}^{2n}, \omega))$ if and only if $\operatorname{rank} Z = n$ and ${}^tXY = {}^tYX$ (i.e. ${}^tXY$ is symmetric).

Proof. It is evident once we notice that the matrix of ω restricted to $\operatorname{Im} Z$ is ${}^tXY = {}^tYX$. □

We call Z as in the lemma a lagrangian frame. The columns of a lagrangian frame $Z = {}^t(XY)$ form a orthonormal basis for L if and only if the matrix $X + iY$ is unitary, and in this case Z is called a unitary lagrangian frame for L. To show that $Gr(Lag(\mathbb{R}^{2n}))$ is a manifold of dimension $n(n-1)/2$, we first note that for a frame Z, if $Y = 0$ then X needs to be symmetric, and the space of symmetric matrices has precisely that dimension. Then, $Z = {}^t(X\,0)$ generates the horizontal Lagrangian

$$\Lambda_{hor} = \left\{ (x, y) \in \mathbb{R}^{2n} \;\middle|\; y = 0 \right\}$$

so that a neighbourhood of Λ_{hor} is identified with the space of symmetric matrices. We conclude by the following lemma:

Lemma 1.2.11. If $\Lambda \in Gr(Lag(\mathbb{R}^{2n}))$, then its image via a symplectic transformation is also lagrangian. Vice versa, for every lagrangian subspace of $\mathbb{R}^{2n}$ Λ there is a symplectic transformation σ such that $\sigma\Lambda = \Lambda_{hor}$. Therefore we also get an isomorphism (and a homeomorphism) $Gr(Lag(\mathbb{R}^{2n})) \simeq U(n)/O(2n)$.

[5]The definition of the Grassmannian can be found in the Appendix, for any smooth manifold.

Via a symplectic transformation we can therefore identify a neighbourhood around any Lagrangian to a neighbourhood around Λ_{hor}, which has a structure as described above.

We notice that identifying $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ the correspondence Lagrangian $\leftrightarrow$ lagrangian frames is a linear analogue of the more general generating functions. In particular, setting $X = \mathbf{1}$ and $Y = A$ for some symmetric matrix A, the lagrangian subspace we have using the frame Z is $\mathrm{Gr}(A) = \{ (x, Ax) \}$. This corresponds to the case where, if we have S and Σ_S as in the definition of generating function, and pr_1 is the projection $\mathbb{R}^n \times \mathbb{R}^k \twoheadrightarrow \mathbb{R}^n$, then $\mathrm{pr}_1(\Sigma_S) = \mathbb{R}^n$ and the lagrangian subspace is given by the linear combinations of the covectors defined by the columns of A, attached to the basis we use to define the matrix.

Since we have the homeomorphism $Gr(Lag(\mathbb{R}^{2n})) \simeq U(n)/O(2n)$, we can assign a Maslov index to paths in $Gr(Lag(\mathbb{R}^{2n}))$ as we did for paths in $Sp(2n)$.

Theorem 1.2.12. The Maslov index for paths in $Gr(Lag(\mathbb{R}^{2n}))$ has the following properties:

Homotopy Two loops in $Gr(Lag(\mathbb{R}^{2n}))$ have the same Maslov index iff they're homotopic.

Product If Λ, Ψ are two loops respectively in $Gr(Lag(\mathbb{R}^{2n}))$ and in $Sp(2n)$, then $\mu(\Psi\Lambda) = \mu(\Lambda) + 2\mu(\Psi)$ (in particular the Maslov index of the constant loop is 0).

Direct sum If $n = n_1 + n_2$, we can identify $Gr(Lag(\mathbb{R}^{2n_1}))$ and $Gr(Lag(\mathbb{R}^{2n_2})))$ as submanifolds of $Gr(Lag(\mathbb{R}^{2n}))$. Then if Λ_j is a path in $Gr(Lag(\mathbb{R}^{2n_j}))$, we have $\mu(\Lambda_1 \oplus \Lambda_2) = \mu(\Lambda_1) + \mu(\Lambda_2)$.

Normalisation If $\Lambda(t) = \exp(\pi i t)\mathbb{R}$ is a path in $Gr(Lag(\mathbb{R}^2))$, then $\mu(\Lambda) = 1$.

Proof. We define the map $\rho : Gr(Lag(\mathbb{R}^{2n})) \to \mathbb{S}^1$ using

$$\rho(\Lambda) = \det(U^2), \quad \text{for } \Lambda = \mathrm{Im} \begin{pmatrix} X \\ Y \end{pmatrix} \text{ and } U = X + iY$$

The frame we chose clearly needs to be unitary. One then proceeds in the same way we did for the Maslov index in $Sp(2n)$. ☐

As the previous index, this one is uniquely defined by the above axioms too. In particular, we can find an analogue interpretation: if $\Lambda_{vert} = \mathrm{Im} \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$ is the vertical Lagrangian (which is simply $\mathbb{R}^n$ seen as the 0-section of its cotangent bundle), we define $\Sigma(n)$ to be the subset of lagrangian subspaces which do not intersect Λ_{vert} transversally. The Maslov index of a path in $Gr(Lag(\mathbb{R}^{2n}))$ is given by the number of (oriented) intersections between such a path and $\Sigma(n)$. In particular, for a path Λ associated to a unitary frame $U(t) = X(t) + iY(t)$, t is said to be a crossing time of the path with $\Sigma(n)$ if and only if $\det(X(t)) = 0$: clearly this happens if and only if $\Lambda(t)$ intersects Λ_{vert} non transversally.

As above we define the notion of regular crossing: given the form $\Gamma(\Lambda, t) : \ker X(t) \to \mathbb{R}$, $\Gamma(\Lambda, t)(u) = \langle \dot{X}(t)u | Y(t)u \rangle$, t is regular if and only $\Gamma(\Lambda, t)$ is non degenerate. Then for a regular crossing we consider the signature $\text{sign}\Gamma(\Lambda, t)$ and define the Maslov index of the path

$$\mu(\Lambda) = \sum_t \text{sign}\Gamma(\Lambda, t)$$

where the sum is taken over the regular crossings. The Maslov index defined this way satisfies the same axioms as in Theorem 1.2.12, so it coincides with the Maslov index we defined above by uniqueness.

On a general symplectic manifold

In this document we justify why the vanishing of $2c_1(TM)$ is a sufficient condition to define a Maslov index, at least up to the (additive) action of a subgroup of the integers. This will follow the paper [33]. Remember that the manifolds are assumed paracompact: this technical requirement is going to be important in some of the proofs.

Let (M, ω) be a symplectic manifold, and L_0, L_1 two lagrangian submanifolds with transverse intersections. Let a, b two points of intersection, γ_i a path in L_i joining a to b be two homotopic curves. Let us then consider a disk $f : \mathbb{D} \to M$ whose boundary is the loop $\gamma = \gamma_1\gamma_2^{-1}$. The bundle f^*TM is trivial, $\mathbb{D}$ being contractible, and all trivialisations are homotopic. We can then suppose $f^*TM = \mathbb{D} \times (\mathbb{R}^{2n}, \omega_0)$ (ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$). We can then build a path in $Gr(Lag(f^*TM))$ as follows: along γ_1 we consider the path $\tau_1(t) = (\gamma_1(t), T_{\gamma_1(t)}L_1)$, whereas on γ_2 we take a path of transverse subspaces; more specifically, we consider $\tilde{\tau}_2$ such that $\tilde{\tau}_2(0) = T_aL_1$, $\tilde{\tau}_2(1) = T_bL_1$, and $\forall t \in I$, $\tilde{\tau}_2(t) \pitchfork T_{\gamma_2(t)}L_2$. This expedient is necessary to have a loop $\tau = \tau_1\tilde{\tau}_2^{-1}$ as desired.

Let $[\mu] \in H^1(Gr(Lag(\mathbb{R}^{2n})); \mathbb{Z})$ be the assignation to a loop of its Maslov index as we defined it. It does define a cohomology class: by Hurewicz Theorem, $\mathbb{Z} \simeq \pi_1(Gr(Lag(\mathbb{R}^{2n}))) \simeq H_1(Gr(Lag(\mathbb{R}^{2n})); \mathbb{Z})$, and by the Universal Coefficient Formula $H^1(Gr(Lag(\mathbb{R}^{2n})); \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(Gr(Lag(\mathbb{R}^{2n}))); \mathbb{Z})$, so it suffices to define μ on the loops. Moreover it is clearly closed, since the image of a disk is contractible and its boundary necessarily nullhomotopic.

Since f^*TM is trivial, we also have a class $[\bar{\mu}] \in H^1(Gr(Lag(f^*TM)); \mathbb{Z})$ which induces $[\mu]$ on the fibres: just take the assignation of the Maslov index to the second projection of a loop in f^*TM. We can therefore define the quantity

Definition 1.2.5. $m(a, b, f) = \langle [\bar{\mu}], \tau \rangle \in \mathbb{Z}$

By definition (and property "homotopy" of the Maslov index as we defined it), $m(a, b, f) = 0$ iff the the path τ is the boundary of a circle in $Gr(Lag(f^*TM))$, so $[\bar{\mu}]$ is in fact an obstruction class. A priori, $m(a, b, f)$ depends on the choices of the paths γ_i, of the application f and of the path $\tilde{\tau}_2$. We shall show how, under the hypothesis of $2c_1(TM) = 0$, we can ignore the dependence on the choice of f (that is, how we cap the loop) and τ (so on the choice of the path of transverse spaces), and control the one on the choice of the loops γ_i.

Let us suppose now to have two loops, γ the same one as before and $\eta = \eta_1\eta_2^{-1}$. Let $\xi = \xi_1\tilde{\xi}_2$ be the analogue for η of τ. Let $g : \mathbb{S}^1 \times I \to M$ be a homotopy between the two cappings: $g(-1,\cdot) = a$, $g(1,\cdot) = b$, such on the upper (resp. lower) half of $\mathbb{S}^1$ it interpolates between γ_1 and γ_2^{-1} (resp. between η_1 and η_2^{-1}). Since $H^2(\mathbb{S}^1 \times I;\mathbb{Z}) = 0$, the vector bundle g^*TM is trivial[6], and one can find here too a cohomology class $[\bar{\mu}(g)] \in H^1(Gr(Lag(g^*TM));\mathbb{Z})$ which induces $[\mu]$ fibrewise. We have therefore the equality

$$m(a,b,f) - m(a,b,f') = \langle[\bar{\mu}(g)],\tau\rangle - \langle[\bar{\mu}(g)],\xi\rangle \tag{1.2}$$

which is clear given the triviality of g^*TM. Let us remark that $[\bar{\mu}(g)]$ is defined up to addition of a cohomology class $\pi^*[\alpha]$, for $[\alpha] \in H^1(\mathbb{S}^1 \times I;\mathbb{Z})$, $\pi : Gr(Lag(g^*TM)) \to \mathbb{S}^1 \times I$ the projection, namely $[\bar{\mu}(g)] + \pi^*[\alpha]$ still induces $[\mu]$ on the fibres. Despite this however, the difference in Equation 1.2 is independent of the choice of α, since the two classes $\pi_*([\tau])$, $\pi_*([\xi])$ coincide.

We can now prove that the difference in Equation 1.2 does not depend on the choice of $\tilde{\tau}_2$ and $\tilde{\xi}_2$. By Hurewicz Theorem $\langle[\bar{\mu}(g)],\tau\rangle - \langle[\bar{\mu}(g)],\xi\rangle = \langle[\bar{\mu}(g)],\tau\xi^{-1}\rangle$, and one can check, always by Hurewicz Theorem, that

$$[\tau\xi^{-1}] = [(\tau_1\xi_1^{-1})(\tilde{\tau}_2\tilde{\xi}_2^{-1})^{-1}] \tag{1.3}$$

We now replace $\tilde{\tau}_2, \tilde{\xi}_2$ with τ_2, ξ_2 which simply are the paths of tangent spaces:

$$\tau_2(t) = T_{\gamma_2(t)}L_2,\ \ \xi_2(t) = T_{\eta_2(t)}L_2$$

It is a classical result that for two transverse lagrangian subspaces of $\mathbb{R}^{2n}$ there is a matrix in $Sp(2n)$ taking one onto the other bijectively. In particular we can replicate the operation pointwise on $\tau_2(t) \pitchfork \tilde{\tau}_2(t)$, $\xi_2(t) \pitchfork \tilde{\xi}_2(t)$ to find a homotopy between the two loops $\tau_2\xi_2^{-1}$ and $\tilde{\tau}_2\tilde{\xi}_2^{-1}$. Given this homotopy and the identity 1.3 we have the following result:

Proposition 1.2.13. *$m(a,b,f) - m(a,b,f') = \langle[\bar{\mu}(g)],(\tau_1\xi_1)(\tau_2\xi_2^{-1})^{-1}\rangle$*

Let us now assume that $2c_1(M) = 0$, where $c_1(M)$ is the first Chern class of TM endowed with an almost complex structure compatible with ω (the space of such structures being contractible, the result in cohomology does not depend on the choice we make). One can prove[7] that if $2c_1(M) = 0$ we can find a cohomology class $[\mu] \in H^1(Gr(Lag(TM));\mathbb{Z})$ inducing the Maslov class on every fibre, and $[\bar{\mu}(g)]$ we considered before is its restriction in the sense that if $e : Gr(Lag(g^*TM)) \hookrightarrow Gr(Lag(TM))$ is the canonical (and continuous) injection, $e^*[\bar{\mu}] = [\bar{\mu}(g)]$. Moreover, let $\phi_i : L_i \to Gr(Lag(g^*TM))$ be the Gauss map $p \mapsto T_pL_i$; we define then $[\bar{\mu}(L_i)] = (e\phi_i)^*[\bar{\mu}] \in H^1(L_i;\mathbb{Z})$.

[6]A symplectic vector bundle on the circle is always trivial, since the symplectic group is connected.

[7]See appendices.

Corollary 1.2.14. If $2c_1(M) = 0$, $m(a, b, f) - m(a, b, f') = \langle[\bar{\mu}(L_1)], [\gamma_1\eta_1^{-1}]\rangle - \langle[\bar{\mu}(L_2)], [\gamma_2\eta_2^{-1}]\rangle$

Proof. It is an easy calculation:

$$m(a, b, f) - m(a, b, f') = \langle[\bar{\mu}(g)], [(\tau_1\xi_1^{-1})(\tau_2\xi_2^{-1})^{-1}]\rangle =$$
$$= \langle[\bar{\mu}(g)], [\tau_1\xi_1^{-1}]\rangle - \langle[\bar{\mu}(g)], [\tau_2\xi_2^{-1}]\rangle = \langle[\bar{\mu}], e_*[\tau_1\xi_1^{-1}]\rangle - \langle[\bar{\mu}], e_*[\tau_2\xi_2^{-1}]\rangle =$$
$$= \langle[\bar{\mu}], e_*\phi_{1*}[\gamma_1\eta_1^{-1}]\rangle - \langle[\bar{\mu}], e_*\phi_{2*}[\gamma_2\eta_2^{-1}]\rangle = \langle[\bar{\mu}(L_1)], [\gamma_1\eta_1^{-1}]\rangle - \langle[\bar{\mu}(L_2)], [\gamma_2\eta_2^{-1}]\rangle$$

□

Remark. If $M = T^*X$, then $2c_1(M) = 0$ automatically. The reason is the following: writing $\pi : T^*X \to X$, and identifying X with the 0-section of T^*X, we have an isomorphism $T(T^*X) \simeq \pi^*T|_X(T^*X)$, where $T|_XT^*X$ is the restriction to $X = 0_{T^*X}$ of the tangent bundle $T(T^*X)$; the reason is that taking a trivialisation around any point we can move tangentially either to the zero section or to the fibres, and the result is the fibre product obtained by pulling back $T|_X(T^*X)$ along π. Moreover, we have an isomorphism $T|_X(T^*X) \simeq TX \otimes_{\mathbb{R}} \mathbb{C}$ (explained below), and $TX \otimes \mathbb{C} \cong \overline{TX \otimes \mathbb{C}}$ given the decomposition $TX \otimes \mathbb{C} \cong T^{(1,0)}X \oplus T^{(0,1)}X$ and since $T^{(0,1)}X \cong \overline{T^{(1,0)}X}$ and viceversa. As a consequence, the total Chern classes need to coincide: $c(TX \otimes \mathbb{C}) = c(\overline{TX \otimes \mathbb{C}})$. However, it is a general fact that $R^{\overline{E}} = -R^E$, where R^E is the curvature of the Chern connection on the complex vector bundle E, and $\overline{E}$ is the conjugated bundle of E; this can be easily seen using that the Chern connection is antisymmetric with respect to the hermitian product on E and the formula

$$R^E(X, Y) = \nabla^E_X\nabla^E_Y - \nabla^E_Y\nabla^E_X - \nabla^E_{[X,Y]}$$

Therefore $c_1(TX \otimes \mathbb{C}) = c_1(\overline{TX \otimes \mathbb{C}}) = -c_1(TX \otimes \mathbb{C})$, hence $2c_1(TX \otimes \mathbb{C}) = 0$. By naturality of the Chern class,

$$2c_1(T(T^*X)) = 2c_1(\pi^*T|_X(T^*X)) = \pi^*2c_1(TX \otimes \mathbb{C}) = 0$$

For the sake of clarity, we recall that a Chern class is defined for a complex vector bundle, which does not need to be holomorphic (indeed $TX \otimes \mathbb{C}$ is not). We shall now explain an isomorphism between $T|_X(T^*X)$ and $TX \otimes \mathbb{C}$. Let $x \in X = 0_{T^*X}$. Fix any almost complex structure J calibrated by ω on T^*X. We obtain a further decomposition $T|_X(T^*X) \simeq TX \oplus J(TX)$: in fact both TX and $J(TX)$ are lagrangian by J-invariance of ω, hence each has a half of the total dimension, and the intersection needs to be trivial since $\omega(\cdot, J\cdot)$ is a positive-definite scalar product. By definition we also have $TX \otimes \mathbb{C} = TX \oplus iTX$, and we can identify the two decomposition making J act as i does on the fibres.

Remark. If we do not suppose $2c_1(M) = 0$ we can still say something about the dependence on the choice of the capping: in particular for two different cappings f, f' for the loop $\gamma_1\gamma_2^{-1}$ we have $m(a, b, f) - m(a, b, f') = 2\langle c_1(M), [\sigma]\rangle$ where

$\sigma \in \pi_2(M)$ is the sphere obtained gluing f and f' along their boundaries. Note that in the case $\langle c_1(M), \pi_2(M)\rangle = 0$, which is a topological assumption one can make to define a Hamiltonian Floer Homology of the manifold, the index does not depend on the choice of the capping.

Corollary 1.2.14 in particular implies that if $2c_1(M) = 0$ we can successfully define a Maslov index which does not depend on the homotopy classes of loops we choose, but this up to the action of a subgroup of $\mathbb{Z}$. In particular, this subgroup of $\mathbb{Z}$, assuming arc-connectedness of the lagrangian submanifolds L_1, L_2, does not depend on the pair of intersection points we are considering. Lastly, if the Maslov classes $[\bar{\mu}(L_i)]$ vanish or if the two lagrangian submanifolds L_i are simply connected, the Maslov index of these loops is well defined, and we can define properly Maslov indices for intersection points too.

Chapter 2

Generating Functions

Main sources for this chapter are [8] and [30]. The aim of this chapter is to introduce the notion of generating function: it is a way to represent via a function a Lagrangian submanifold of a cotangent bundle. While its usefulness will become evident later on, this part of the thesis is devoted to technical results which is necessary to provide a context where all the homologic machinery will work.

2.1 Basic definitions

Let M be an n-dimensional, smooth, connected manifold. Let (T^*M, ω) be its cotangent bundle endowed with its standard symplectic form $\omega = -d\lambda$.

Definition 2.1.1 (Symplectic isotopy)**.** A symplectic isotopy in (T^*M, ω) is a path of symplectic diffeomorphisms $(\varphi_t)_{t\in I}$ such that $\varphi_0 = Id$, and that if $X_t = \frac{d}{dt}\varphi_t$, then $L_{X_t}\omega = 0$.

Definition 2.1.2 (Hamiltonian isotopy)**.** A symplectic isotopy is Hamiltonian if there is a smooth function $H \in \mathcal{C}^\infty(I \times T^*M)$ such that $dH_t = \iota_{X_t}\omega$

Notice that for any Hamiltonian $H \in \mathcal{C}^\infty(I \times T^*M)$, setting $\varphi_t = \phi_H^t$, gives a Hamiltonian isotopy, as it trivially satisfies $dH_t = \iota_{X_t}\omega$, which in turn implies $L_{X_t}\omega = 0$ by Cartan's formula.

Definition 2.1.3 (Exact Lagrangian submanifold)**.** An embedded Lagrangian submanifold (whose tangent space is Lagrangian at every point) $e : L \to T^*M$ is exact if $e^*\lambda$ is exact.

Lemma 2.1.1. The image under a symplectic isotopy of a Lagrangian submanifold $e : L \to T^*M$ is Lagrangian. The image under a Hamiltonian isotopy of an exact Lagrangian submanifold $e : L \to T^*M$ is exact Lagrangian.

Proof. Let φ_t be the symplectic, or Hamiltonian in the second part of the proof, isotopy. Denote $e_t = \varphi_t \circ e$ the embedding for $\varphi_t(L)$.

In the first part of the proof, we show that $e_t^*\omega = e^*\omega = 0$, showing that the result of the deformation is indeed Lagrangian. But

$$\frac{d}{ds}_{\restriction s=t} e_s^*\omega = \frac{d}{ds}_{\restriction s=t} e^*\varphi_s^*\omega = \frac{d}{ds}_{\restriction s=t} e^*\omega = 0$$

and the result follows. The second equality comes from the simplecticity of $(\varphi_t)_t$.

For the second part, again by the Fundamental Theorem of Calculus,

$$e_t^*\lambda - e^*\lambda = \int_0^t \frac{d}{ds} e_s^*\lambda ds$$

and we need to show that the integrand is exact. We compute its derivative:

$$\frac{d}{dr}_{\restriction r=s} e^*\varphi_r^*\lambda = e^*\varphi_s^* L_{X_s}\lambda = e_s^*(d\iota_{X_s}\lambda - \iota_{X_s}\omega) = de_s^*(\iota_{X_s}\lambda - H_s)$$

We used here that the pull-back commutes with d and the definition of H_t.We conclude that $\varphi_t(L)$ is exact Lagrangian:

$$e_t^*\lambda = d\left\{\int_0^t (e_s^*\iota_{X_s}\lambda - e_s^* H_s)ds + f\right\}$$

for some $f \in \mathcal{C}^\infty$, since $e^*\lambda$ is exact. □

We now introduce generating functions and forms in the vector bundle setting: let $E \xrightarrow{\pi} M$ be a vector bundle on M. We define its *coisotropic subbundle* as

$$W_e = \left\{ \xi \in T_e^*E \;\middle|\; e \in E,\, \xi|_{T_e(\pi^{-1})\pi e} = 0 \right\} \tag{2.1}$$

In other words, W is the orthogonal bundle to $d\pi^{-1}(0_{TM})$ (0_{TM} is the zero section of the tangent bundle): it is indeed a subbundle of T^*E.

Definition 2.1.4 (Generating form)**.** A closed 1-form $\alpha \in Z^1(E)$ is a generating form if, seen as a section in $\Gamma(T^*E)$, is transverse to W.

Definition 2.1.5 (Generating function)**.** A function $S : E \to \mathbb{R}$ is a generating function if dS is a generating form.

Certain Lagrangian submanifolds of T^*M are determined by generating forms or functions, in the following way: if Σ_α (resp. Σ_S) is the critical locus $\alpha^{-1}(W)$ (resp. $d^vS^{-1}(W)$), hence by transversality a submanifold of T^*E of dimension $\dim M$, one can define a Lagrangian immersion

$$i_\alpha : \Sigma_\alpha \to T^*M$$

that maps a point $e \in \Sigma_\alpha$ to the covector

$$T_{\pi(e)}M \ni v \mapsto \alpha_e(\tilde{v}) \in \mathbb{R}$$

where $\tilde{v}$ is any lift of v via $d_e\pi$. The good definition is obvious, as α vanishes by construction on the difference of such two lifts. The definition of i_α includes the case $\alpha = dS$ of generating functions.

Lemma 2.1.2. i_α is a Lagrangian immersion.

Proof. Let us start by showing that i_α is an immersion. For the sake of simplicity we consider the case $\alpha = dS$, the non exact case is the same. The statement is local, so that we suppose the vector bundle to be trivial: $\pi : E = M \times \mathbb{R}^k \to M$. Let $\eta \in T_{(x,v)}\Sigma_S$, $\eta \in \ker d_{(x,v)} i_S$. Let us consider a curve $(x(t), v(t)) \in \Sigma_S$ representing η. Then since the first coordinate of $i_S(x(t), v(t))$ is $x(t)$, we obtain that $\dot{x}(0) = 0$, and $\eta \in \ker d_{(x,v)}\pi$ is vertical. η being tangent to Σ_S, this implies that $\dot{v}(0) \in \ker d^\lambda_{(x,v)} \partial^\lambda S$. In the trivial bundle case, $W = M \times \{0\}$: writing down the definition of the transversality condition, one sees that this is equivalent to asking that $d^\lambda_{(x,v)} \partial^\lambda S$ is an isomorphism $\mathbb{R}^k \to \mathbb{R}^k$: this concludes the proof.

Let us prove i_α is Lagrangian. Fix some $e \in E$, r a local section of π such that $r\pi(e) = e$, and denote $\mathrm{pr} : TM \to M$ the canonical projection. Then for any $v \in T_e E$, we locally have

$$(i_\alpha^* \lambda)_e(v) = \alpha_e(d_e(r \circ \mathrm{pr} \circ i_\alpha)v) = (i_\alpha^* \mathrm{pr}^* r^* \alpha)_e(v)$$

Since α is closed d is local and commutes with the pull-backs, and $i_\alpha \Sigma_\alpha$ has dimension $\dim M$, we establish the result. Notice moreover that in the case of a generating function, the immersion is exact Lagrangian. □

Definition 2.1.6. We say that a Lagrangian submanifold $L \leq T^*M$ is generated by a generating form (resp. function) α (resp. S) if i_α (resp. i_S) is a diffeomorphism onto L.

The exact case has a more explicit description: $S : M \times \mathbb{R}^k \to \mathbb{R}$ is a generating function for the lagrangian submanifold $L \leq T^*M$ if, denoting with $\partial^\lambda S$ the vertical derivative,

i) $\partial^\lambda S \pitchfork 0$

ii) $L = \left\{ \xi \in T^*M \;\middle|\; \partial^\lambda S(\pi(\xi), \lambda) = 0, \exists \lambda,\ \xi = (\pi(\xi), d^x_{\pi(\xi)} S(\cdot, \lambda)) \right\}$

2.1.1 The equivalence relation

To state the main theorems we are going to prove, we need to introduce some notions: a non-degenerate quadratic form $Q : E \to \mathbb{R}$ is a function which is a non-degenerate quadratic form when restricted to the fibres. We can then define the generating forms (and functions) which are *quadratic at infinity* (QI): a generating form α is QI if there is a non-degenerate quadratic form Q such that $\alpha - \partial Q : E \to E^*$ is bounded. ∂Q here is the fibrewise derivative of Q. Analogously, a generating function S is QI if its differential dS is. If moreover $S = Q$ (for a non-degenerate quadratic form Q) outside a compact set, S is *exactly* QI. An exactly QI generating function is *special* if the bundle on which it is defined is trivial, and the non-degenerate quadratic form does not depend on the point of the manifold.

Generating functions and forms for a fixed Lagrangian submanifold of T^*M are certainly not unique, in a strict sense: the following operations in fact produce from a generating function another one, which determines the same manifold as the first one.

Definition 2.1.7 (Basic operations)**.** Let $E \xrightarrow{\pi} M$ be a vector bundle, α a generating form defined on E, and S a generating function. Then we define the following three operations:

- If $c \in \mathbb{R}$, we define $S' = S + c$ (Addition of a constant);
- If $E' \xrightarrow{\pi'} M$ is another vector bundle and $\Phi : E' \to E$ is an fibre-preserving diffeomorphism, we define $\alpha' = \Phi^*\alpha$, $S' = \Phi^*S$ (Diffeomorphism operation);
- If $E' \xrightarrow{\pi'} M$ is another vector bundle, with a non-degenerate quadratic form Q', we define $\alpha' = \alpha \oplus dQ'$, $S' = S \oplus Q'$ (Stabilisation).

Definition 2.1.8. Two generating forms or functions are equivalent if one of them is obtained after a finite number of basic operations from the other.

We remark that the three operations commute, in a strict sense: different orders determine the same generating forms or functions (not just equivalent ones!); it might be necessary to extend the diffeomorphisms by the identity on the second vector bundle.

Lemma 2.1.3. The image of a generating form α or function S under a basic operation generates the same Lagrangian submanifold of T^*M as α and S.

Proof. Clearly adding a constant to the function S does not change the differential, so this part is trivial. We now prove the invariance under fibre-preserving diffeomorphisms; the direct sum will be similar.

Let $W' \leq T^*E'$ be the coisotropic subbundle of E'. Then an easy calculation shows that $\xi \in W$ if and only if $\Phi^*\xi \in W'$. A similar argument shows that $e' \in (\Phi^*\alpha)^{-1}(W')$ if and only if $e \in \alpha^{-1}(W)$: this means that $\Sigma_{\Phi^*\alpha} = \Phi^{-1}(\Sigma_\alpha)$. Then, for $e' = \Phi^{-1}(e) \in \Sigma_{\Phi^*\alpha}$, the associated covector is correctly in $T^*_{\pi'(e')}M = T^*_{\pi'(\Phi^{-1}e)}M = T^*_{\pi(e)}M$, and it acts the following way: if $v \in T^*_eM$ has a lift $\tilde{v}$ for $d_e\pi$, then

$$d_{e'}\pi'.(d_e\Phi^{-1}.\tilde{v}) = d_e(\pi' \circ \Phi^{-1}).\tilde{v} = d_e\pi.\tilde{v} = v$$

i.e. $d_e\Phi^{-1}.\tilde{v}$ is a lift of v for $d_e\pi'$, hence

$$i_{\Phi^*\alpha}(e')(v) = \alpha_{\Phi e'}(d_{e'}\Phi \circ d_e\Phi^{-1}.\tilde{v}) = \alpha_e(\tilde{v}) = i_\alpha(e)(e)$$

which is what we wanted to show. □

Proposition 2.1.4. Any generating function QI is equivalent to a special one.

Proof. Via stabilisation, we can always suppose to be working with a trivial bundle: there exists a vector bundle E' such that $e \oplus E'$ is trivial (see [6], Theorem 14.2 for a proof). Stabilise using E endowed with any non-degenerate form. From now on we assume the vector bundle to be trivial $E = M \times \mathbb{R}^k$. The next step to prove is that one can always assume that the quadratic form does not depend on the point of the base M: indeed, let E^+, E^- be the subbundles of E where Q is positive, or negative, definite. These bundles may not be trivial, but their direct sum is. If they are, we can apply Gram-Schmidt fibrewise separately, finding a diffeomorphism $\Phi : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ such that $\Phi^* Q$ has the desired property. If they are not trivial, we can stabilise the (E, S) (S associated to the quadratic form Q) with $(E, -Q)$: since $-Q$ has stable and unstable spaces swapped with respect to those of Q, the two subbundles where $Q \oplus -Q$ is negative or positive definite are trivial: applying Gram-Schmidt then gives us a further diffeomorphism $\Phi' : M \times \mathbb{R}^{2k} \to M \times \mathbb{R}^{2k}$ such that $\Phi'^*(Q \oplus -Q)$ does not depend on the point on the base space.

So far we have shown that every generating function QI is equivalent to another one defined on a trivial vector bundle on M, where the quadratic form does not depend on the coordinates on M. We still need to prove that S is exactly QI. Look at the doctoral dissertation of David Théret [31] for further details. □

We are first going to prove the following Theorem:

Theorem 2.1.5 (Sikorav)**.** If L is a Lagrangian submanifold of T^*M, for a closed manifold M, with a generating function and $(\varphi_t)_t$ is a Hamiltonian isotopy, then $\varphi_t(L)$ also has a generating function.

The next Theorem is a first result of uniqueness:

Theorem 2.1.6 (Viterbo)**.** If L is a Hamiltonian deformation of the zero section in T^*M, then its generating function is unique, up to equivalence.

We are going to prove that these results still hold true when the deformation is only symplectic, and not Hamiltonian.

In the exact case, we are going to prove that the uniqueness property is stable under Hamiltonian isotopies, and that it already holds for the zero section (remark that the GFQI for the zero section are just non-degenerate forms, so it is an easier problem to approach). To extend to the Symplectic case, we shall show that every Lagrangian submanifold can be deformed into an exact Lagrangian one, and that up to homotopy the symplectic deformations can be assumed Hamiltonian.

2.2 A proof of Sikorav's Theorem

Brunella proves the Theorem carrying the problem to $\mathbb{R}^N$ using Chekanov's Trick (here the compactness of M is crucial to apply Whitney's Theorem), where he solves it.

2.2.1 First Step: Chekanov's Trick

Aim: carry the abstract problem into $T^*\mathbb{R}^N$, for some N.

Let then M be a compact manifold. By Whitney's Theorem we have an embedding $i : M \hookrightarrow T^*\mathbb{R}^N$ for some $N > 0$. This induces an embedding $Ti := (i, di) : TM \hookrightarrow T\mathbb{R}^N$. If we fix two Riemannian metrics, one on M and one on $\mathbb{R}^N$ such that i is an isometry[1] we can identify $TM \simeq T^*M$ and $T\mathbb{R}^N \simeq T^*\mathbb{R}^N$ via two diffeomorphisms ζ_M, $\zeta_{\mathbb{R}^N}$. We can then define $j = \zeta_{\mathbb{R}^N} \circ Ti \circ \zeta_M^{-1} : T^*M \hookrightarrow T^*\mathbb{R}^N$, which is symplectic.

2.2.2 Second Step: Translation, Solution of the problem in $\mathbb{R}^N$

Here we will state (without proof) three lemmas which will on one hand explain what kind of relation we can find between L of the statement and $j(L)$, and on the other hand how to solve the problem for $j(L)$.

We need at first observe that the decomposition $T\mathbb{R}^N|_{i(M)} = (Ti)(TM) \oplus N_{i(M)}$ ($N_{i(M)}$ is the normal bundle of $j(TM)$ in $T\mathbb{R}^N|_{i(M)}$) induces by duality $T^*\mathbb{R}^N|_{i(M)} = j(T^M) \oplus N^*_{i(M)}$ ($N^*_{i(M)}$ is the conormal bundle).

Lemma 2.2.1. Let $\phi^t : T^*M \to T^*M$ be a Hamiltonian isotopy, and $j : T^*M \to T^*\mathbb{R}^N$ as above. Then there exists a Hamiltonian isotopy $\psi^t : T^*\mathbb{R}^N \to T^*\mathbb{R}^N$ verifying the three following properties:

i) $j \circ \phi^t = \psi^t \circ j$

ii) $\psi^t(T^*\mathbb{R}^N|_{i(M)}) \subseteq T^*\mathbb{R}^N|_{i(M)}$

iii) If V is any neighbourhood of $i(M)$, we can choose ψ^t with support in $T^*\mathbb{R}^N|_V$

This lemma extends ϕ^t to a new Hamiltonian isotopy ψ^t; the meaning of points ii) and iii) will become clearer later on.

Lemma 2.2.2. Let $L \leq T^*M$ be an immersed Lagrangian submanifold, with a GFQI $S : M \times \mathbb{R}^k \to \mathbb{R}$. Then there exists a Lagrangian submanifold of $T^*\mathbb{R}^N$, $\tilde{L}$, with a GFQI $\tilde{S} : \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}$ such that $\tilde{L} \cap (T^*\mathbb{R}^N)|_{i(M)} = j(L)$, and the intersection is transverse. . Moreover, if $V \subset \mathbb{R}^N$ is a neighbourhood of $i(M)$, we may choose $\tilde{L}$ and $\tilde{S}$ with $\tilde{L} = 0$ outside $T^*\mathbb{R}^N|_{i(V)}$ and $\tilde{S} = Q$ (the quadratic function associated to S) outside $V \times \mathbb{R}^k$.

This Lemma is in a way natural after stating the previous one: if Lemma 2.2.1 extends the isotopy, Lemma 2.2.2 extends the Lagrangian submanifold itself (we see it in the condition $\tilde{L} \cap (T^*\mathbb{R}^N)|_{i(M)} = j(L)$). To do so, in the proof we use a tubular neighbourhood $q_0 : W \to i(M)$ of $i(M)$ in $\mathbb{R}^N$ to extend and localise the problem around the immersed manifold, and there define the new "extended" generating function as $\tilde{S}(x, \lambda) = S(i^{-1}q_0(x), \lambda)$.

[1] We can just choose the Euclidean scalar product on $\mathbb{R}^N$ and then pull it back along i.

Lemma 2.2.3. Let $\tilde{L} \leq T^*\mathbb{R}^N$, $L \subseteq T^*M$ be Lagrangian submanifolds, with $j(L) = \tilde{L} \cap (T^*\mathbb{R}^N)|_{i(M)}$ (transversally). If $\tilde{S} : \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}$ is a GFQI for $\tilde{L}$, then $S : M \times \mathbb{R}^k \to \mathbb{R}$, $(x, \lambda) \mapsto \tilde{S}(i(x), \lambda)$ is a GFQI for L.

With Lemma 2.2.3 we know that solving the problem in $T^*\mathbb{R}^N$ is enough to solve it on T^*M. Consider then $L \leq T^*M$, $S : M \times \mathbb{R}^k \to \mathbb{R}$, $\phi^t : T^*M \to T^M$ as in the hypotheses of Theorem 2.1.5. Apply Lemma 2.2.1 to find the ψ^t an Lemma 2.2.2 to extend L to $\tilde{L}$ and S to $\tilde{S}$. Then since ψ stabilises $T^*\mathbb{R}^N|_{i(M)}$ and as ϕ^t and ψ^t are conjugated by j we have

$$j(\phi^1(L)) = \psi^1(j(T^*M)) = \psi^1(\tilde{L}) \cap (T^*\mathbb{R}^N)|_{i(M)}$$

where the intersection is transverse. By Lemma 2.2.3 it suffices then to find a GFQI for $\psi^1(\tilde{L})$. We can decompose ψ^1 in a product

$$\psi^1 = \psi^1 \circ \left(\psi^{\frac{l-1}{l}}\right)^{-1} \circ \cdots \circ \psi^{\frac{i+1}{l}} \circ \left(\psi^{\frac{i}{l}}\right)^{-1} \circ \cdots \circ \left(\psi^0\right)^{-1}$$

and for l large enough we have that $g_i = \psi^{\frac{i+1}{l}} \circ \left(\psi^{\frac{i}{l}}\right)^{-1}$ is $\mathcal{C}^1$-close to the identity (quick calculation; clear if $\psi^t \circ \psi^s = \psi^{t+s}$), and so admits a generating function F_i. Let us remark that to have a uniform estimate in space for the norm $\|g_i - Id\|_{\mathcal{C}^1}$ we need to use the compactness of $\mathrm{Supp}(\psi^t)$ for $t \in I$; this also implies that $\mathrm{Supp}(F_i)$ is compact. Applying the next Lemma to every g_i, we achieve the proof of Theorem 2.1.5.

Lemma 2.2.4. Let $\tilde{L} \leq T^*\mathbb{R}^N$ be an immersed Lagrangian submanifold with GFQI $\tilde{S} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $\tilde{S}(x, \lambda) = Q(\lambda)$ outside a compact set. Let $g : T^*\mathbb{R}^N \to T^*\mathbb{R}^N$ be a symplectic diffeomorphism with a generating function $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ of compact support. Then $g(\tilde{L})$ also has a GFQI.

2.3 Sikorav's result as a Theorem on fibrations

Definition 2.3.1 (Serre fibration)**.** A continuous map $X \to B$, for X and B topological spaces, is a Serre fibration (or simply fibration) if for any n, for any continuous map $I^{n-1} \times \{0\} \to X$ there is a continuous function $I^n \to X$ making the following diagram commute:

$$\begin{array}{ccc} I^{n-1} \times \{0\} & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ I^n & \longrightarrow & B \end{array}$$

We say that $X \to B$ has the homotopy lifting property on the cubes.

We now state more precisely Theorem 2.1.5, as done in [29]. The setting is the same, but the intuition of a Serre fibration will be clearer: now we shall consider a path of Hamiltonian deformations, which we shall lift as a continuous path of generating functions.

Theorem 2.3.1. *Let M be a closed manifold, φ_t be a Hamiltonian isotopy on T^*M, $L \leq T^*M$ a Lagrangian submanifold admitting a special generating function $S : M \times \mathbb{R}^k \to \mathbb{R}$. Then there exists an integer $l \geq 0$ and a path $(S_t)_{t \in I}$ of special generating functions defined on $M \times \mathbb{R}^k \times \mathbb{R}^{2l}$ such that:*

- $\forall x$, $S_t(x)$ *is smooth;*
- S_0 *is obtained by stabilisation of S via a non-degenerate quadratic form of signature l;*
- $S_t = S_0$ *outside a compact set;*
- S_t *is a generating function for $\varphi_t(L)$.*

To be able to properly speak of Serre fibration, we need to have a topology and a notion of differentiability on the set of Lagrangian submanifolds and on that of generating functions. Let then $\mathcal{L}$ be the set of Lagrangian submanifolds of T^*M diffeomorphic to the zero section, and admitting a GFQI (which can then be supposed special). Then the differential structure on $\mathcal{L}$ is induced by those of M and T^*M: for any manifold N, a function $f : N \to \mathcal{L}$ is smooth if there exists a smooth function $\tilde{f} : N \times M \to T^*M$ such that $f(n, \cdot)$ is a Lagrangian embedding, whose image is $f(n)$.

Similarly, we can define a notion of smoothness on the set $\mathcal{F}_k$ of generating functions defined on a vector bundle of rank k (we shall often omit the k) which is compatible with the Whitney Strong Topology (see [15] for the definition; being a generating function is an open condition in $\mathcal{C}^\infty(M \times \mathbb{R}^k)$): for a manifold N, a function $f : N \to \mathcal{F}$ is smooth if it is continuous for the strong topology on $\mathcal{F}$ and there exists a function $\tilde{f} : N \times M \times \mathbb{R}^k \to \mathbb{R}$ such that $\tilde{f}(n, \cdot, \cdot) = f(n)$. The reason to consider the Whitney Strong Topology on $\mathcal{F}$ is that the family of functions whose existence is proved by Theorem 2.3.1 is continuous.

Before stating the precise Theorem about generating functions and Serre fibrations, we are going to prove a technical lemma. Δ_n will be the standard n-simplex in $\mathbb{R}^{n+1}$, and we mention that clearly its cotangent bundle is trivial, Δ_n being contractible: $T^*\Delta_n = \Delta_n \times (\mathbb{R}^n)^*$.

Lemma 2.3.2. *Let $f : \Delta_n \times M \to T^*M$ be a smooth map, such that for every fixed $u \in \Delta_n$, $f(u, \cdot) = f_u$ is an exact Lagrangian embedding. Then there is a map $v : \Delta_n \times M \to (\mathbb{R}^n)^*$ such that*

$$L = \{ \, (u, v(u,x), f(u,x)) \in T^*(\Delta_n \times M) \, \}$$

is an exact Lagrangian submanifold of $T^*(\Delta_n \times M)$. Moreover, the difference of two such maps is the differential of a function $c : \Delta_n \to \mathbb{R}$.

Furthermore, if we are given a smooth family $(S_u : M \times \mathbb{R}^k \to \mathbb{R})_{u \in \Delta_n} \subset \mathcal{F}_k$ which generate $L_u = f_u(M)$, then the function $S : \Delta_n \times M \times \mathbb{R}^k \to \mathbb{R}$, $(u,p) \mapsto S_u(p)$ generates L above.

On the other hand, assume $F : \Delta_n \times M \to T^*(\Delta_n \times M)$ is an exact Lagrangian embedding, and let L be its image. If $W_u = \{u\} \times (\mathbb{R}^n)^* \times T^*M$,

$F \pitchfork W_u$ for all $u \in \Delta_n$, and $S : \Delta_n \times M \times \mathbb{R}^k \to \mathbb{R}$ is a generating function for L, then S_u defined above is a generating function for the Lagrangian submanifold $L_u \leq T^*M$ obtained via symplectic reduction of L with respect to the spaces W_u.

Proof. Assume then that the function $v : \Delta_n \times M \to (\mathbb{R}^n)^*$ in the statement exists. Define the function $F : \Delta_n \times M \to T^*(\Delta_n \times M)$ $F(u,x) = (u, v(u,x), f_u(x))$. It is clearly an embedding, whose image is L: we need then to check that $F^*\lambda_{\Delta_n \times M}$ is exact. With some calculations, remembering as we recalled earlier on that $T^*\Delta_n = \Delta_n \times (\mathbb{R}^n)^*$ is trivial,

$$F^*\lambda_{\Delta_n \times M}(u,x)(\xi,\eta) = v(u,x).\xi + \left(f_u^*\lambda_M\right)_x(\eta) + \lambda_M(f_u(x))(d_u f(\cdot,x).\xi)$$

By hypothesis the f_u are exact embedding by definition: there is a smooth function $\alpha : \Delta_n \times M \to \mathbb{R}$ such that $d\alpha_u = f_u^*\lambda_M$. We are then going to define v in a way that it kills the third term and completes the second one to $d\alpha$:

$$v(u,x) = d_u\alpha(\cdot,x) - \lambda_M(f_u(x))(d_u f(\cdot,x).)$$

and therefore $F^*\lambda_{\Delta_n \times M}(u,x)(\xi,\eta) = d\alpha_{(u,x)}(\xi,\eta)$, which is what we wanted to achieve. Once again, the statement about the generating functions just involves some calculations. Instead, to conclude, assume that v, w are two maps $\Delta_n \times M \to (\mathbb{R}^n)^*$ such that the two associated embeddings are exact. By the same calculations as above, the form $(w-v)(u,x)(\xi)$ is exact: there needs to be a function $c_x : \Delta_n \to \mathbb{R}$ such that $(w-v)(\cdot,x) = dc$.

For the second part, W_u is coisotropic: its orthogonal is contained in $(\mathbb{R}^n)^*$ which is the vertical Lagrangian of $\mathbb{C}^n = T^*\mathbb{R}^n$. Its regularity is clear: the quotient is homeomorphic to T^*M which is Hausdorff (the leaf is $(\mathbb{R}^n)^*$ for all $u \in \Delta_n$), and for the existence of the submanifold verifying the condition of the definition of regular coisotropic submanifold, if $(u,v,p) \in W_u$, one can take $S = \{u\} \times \{v\} \times T^*M$. Since the intersection is transverse by Proposition 1.2.8 the symplectic reduction L_u is indeed a lagrangian submanifold of T^*M. The fact that S_u is a generating function of L_u is then immediate. □

We can now state and prove the announced result:

Theorem 2.3.3. Up to equivalence of generating functions of the lifts, $\pi : \mathcal{F} \to \mathcal{L}$ is a smooth Serre fibration: suppose given a smooth function $f : \Delta_n \times I \to \mathcal{L}$, and a lift $F_0 : \Delta_n \to \mathcal{F}$. Then up to replacing F_0 with an equivalent GFQI, there is a lift $F : \Delta_n \times I \to \mathcal{F}$ for f: for every time t, $\pi(F_t) = f_t$.

Proof. Suppose given $f : \Delta_n \times I \to \mathcal{L}$, and let S be the lift at 0. For every fixed time t, by definition of smoothness for functions with values in $\mathcal{L}$, there is a function $\tilde{f}_t : \Delta_n \times M \to T^*M$ such that for every u $\tilde{f}_{t,u}$ is an exact Lagrangian embedding (we have exactness since we consider Hamiltonian isotopies). By previous lemma then we find a family of exact Lagrangian embedding $\bar{f}_t : \Delta_n \times M \to T^*(\Delta_n \times M)$, which again by definition of smoothness for maps into $\mathcal{L}$ is the same as a smooth map $\bar{f} : I \to \mathcal{L}(T^*(\Delta_n \times M))$. From the lemma we also have the lift at the time 0, $\bar{S}_0$.

The second part of the Lemma states that if we can lift the path $\bar{f}$ starting from $\bar{S}_0$ (or an equivalent function) we are done. Let us admit for the moment that we can apply Theorem 2.3.1 in the setting of manifolds with boundary: we have a smooth path of generating functions lifting $\bar{f}$, which is what we needed.

We can justify the application of Sikorav's Theorem in our case of the manifold with boundary $\Delta_n \times M$ as follows: let N be a manifold with boundary, (f_t) a smooth path of exact Lagrangian embeddings $N \to T^*N$ such that $f_t(\partial N) \subset T^*_{\partial N}N$ (it is verified in our case, the path F_t even fixes the points in Δ_n). The associated Hamiltonian isotopy $(\varphi_t)_{t\in I}$ then verifies $\varphi_t \circ f_0 = f_t$ and $\varphi_t(T^*_{\partial N}N) = T^*_{\partial N}N$. Then we can consider the double of N, $2N = N \coprod_{\partial N} N$, and extend the Hamiltonian isotopy extending the Hamiltonian function, and similarly extend the embedding f_0 and the generating function S_0. We apply the Theorem on $2N$, and then restrict to N the lifts. □

Remark. Although not explicitly stated in the Theorem, since in the proof we make use of the fact that $f_{t,u}$ are exact Lagrangian embeddings, we do require the smooth functions $f : \Delta_n \times I \to \mathcal{L}$ to define Hamiltonian isotopies.

2.4 A proof of Viterbo's Theorem: the exact case

2.4.1 Uniqueness under isotopies

We are going to prove one of the two statements necessary towards the proof of Viterbo's Uniqueness Theorem:

Theorem 2.4.1. Assume L is a Lagrangian submanifold of T^*M having the uniqueness property (i.e. all its generating functions are equivalent). Then if (φ_t) is a Hamiltonian isotopy, also $L_1 = \varphi_1(L)$ has it.

The proof of this is divided into two: at first, we proof a property of path-connectedness of $\mathcal{F}$, and then that the endpoints of paths in $\mathcal{F}$ within a certain class are equivalent.

Lemma 2.4.2. Let S, S' be two generating functions for L_1. Then they can be connected by a path contained in $\pi^{-1}(L_1)$, where $\pi : \mathcal{F} \to \mathcal{L}$ is the weak Serre fibration we defined in Section 2.3.

Proof. By assumption, there is a smooth path in $\mathcal{L}$, $(L_t)_{t\in I}$, linking L_0 to L_1, induced by the Hamiltonian isotopy $\varphi : I \to \mathcal{L}$, $t \mapsto \varphi_t(L_0)$. We can then lift the homotopy starting at the time 1, with starting (more precisely, ending) points S and S'. We find two paths $(S_t)_{t\in I}$, $(S'_t)_{t\in I}$ such that S_1 and S'_1 are equivalent to respectively S and S'. By definition, S_0 and S'_0 are generating functions for L_0, which has the uniqueness property: we can make basic operations on the whole path and assume therefore $S_0 = S'_0$.

If we consider the loop γ in $\mathcal{L}$ based at L_1 obtained by following backwards the isotopy and then in the right direction, we have the lift $\tilde{\gamma}$ given by following the path $(S_t)_{t\in I}$ in reverse direction, and then $(S'_t)_{t\in I}$. However, γ is clearly homotopic to the constant loop at L_1: there is a homotopy $H : I^2 \to \mathcal{L}$ such that $H(t,0) = \gamma(t)$, $H(t,1) = L_1$. The homotopy also satisfies the Hamiltonian requirement, since it consists basically in following γ in reverse time: we can lift it, and find a smooth homotopy $\tilde{H} : I^2 \to \mathcal{F}$ with the starting condition $\tilde{H}(t,0) = \tilde{\gamma}(t)$. Then $\tilde{H}(t,1)$ is a path in $\pi^{-1}(L_1)$, and $\tilde{H}(0,1) = S$, $\tilde{H}(1,1) = S'$. □

To finish the proof of the Theorem, we prove the following fact:

Lemma 2.4.3. If (S_t) is a path of generating functions generating the same Lagrangian submanifold, S_0 and S_1 are equivalent.

It is then clear that the uniqueness property will also hold for $L_1 = \varphi(L)$.

Proof. Assume that the generating functions are defined on $E = M \times \mathbb{R}^k$: we are going to look for a family of fibrewise diffeomorphism Φ_t, such that for all t

$$S_t(\Phi_t(x,v)) = S_0(x,v) \tag{2.2}$$

If we achieve it, of course the two generating functions will be equivalent under a *diffeomorphism operation*. Denote via ψ_t the diffeomorphism induced on the fibres: $\Phi_t(x,v) = (x, \psi_t(x,v))$ (it needs not be linear). For every $(x,v) \in M \times \mathbb{R}^k$ we define the vector field $X_t(x,v) = \frac{d}{dt}\psi_t(x,v)$. We derive the identity (2.2) over t: we shall try to solve the equation

$$\partial_t S_t(x,v) + \partial_v S_t(x,v).X_t(x,v) = 0$$

where $\partial_v S(x,v)$ denotes the vertical differential of S. Notice that if Σ_t is the preimage along dS_t of its critical locus, and $i_t : \Sigma_t \to T^*M$ the Lagrangian immersion, then outside a neighbourhood U of Σ_t in $M \times \mathbb{R}^k$ the length of $\partial_v S$ is greater then a strictly positive constant: to solve the equation outside U it suffices to set

$$X_t(x,v) = -\frac{-\partial_t S_t(x,v)}{\|\partial_v S_t(x,v)\|^2}\partial_v S_t(x,v)$$

Solving it on U requires the following fact (for a proof, see [31]): up to fibre isotopies and addition of constants, we can assume that Σ_t and i_t do not depend on t, and $\partial_t S_t = 0$ on Σ_t. Therefore, around $\Sigma = \Sigma_t$, we can apply Hadamard's Lemma for $\partial_t S_t$ and $\partial_v S$: at every $(x,v) \in \Sigma$, there exists a neighbourhood in $\mathbb{R}^k$ of v, $U_{x,v}$, and a function $X_t(x,\cdot) : U_{x,v} \to \mathbb{R}^k$ such that

$$\forall w \in U_x, \ \ \partial_t S_t(x,w) = -\partial_v S(x,v) X_t(x,w)$$

Set $U = \bigcup_{(x,v)\in\Sigma} U_{x,v}$ (remark that U is open, since Σ is a submanifold of $M \times \mathbb{R}^k$ without boundary), and $X_t(x,w) = 0$ on this neighbourhood: we can then use a bump function to glue the two local solutions we found, and define a global one for the differential equation. □

2.4.2 The uniqueness for the 0 section

To finish the proof of Viterbo's Uniqueness Theorem, we only need to prove the following Theorem, for obvious reasons:

Theorem 2.4.4. Let M be a closed manifold. Then the GFQI of the zero section of T^*M are all equivalent.

We start by recalling that every GFQI is equivalent to a special one, as we proved, and therefore we can limit our discussion to this frame. Assume that then $S : M \times \mathbb{R}^k \times \mathbb{R}$ is a special generating function for 0_{T^*M}, associated to a non-degenerate quadratic form Q_∞ which does not depend on the base point of the vector bundle. Let $i_S : \Sigma_S \to T^*M$ be the Lagrangian immersion associated to S.

By definition of i_S, since S generated the zero section, if $\pi : E = M \times \mathbb{R}^k \to M$ is the bundle projection, $\pi(\Sigma_S) = M$: Σ_S is the graph of a map $v_S : M \to \mathbb{R}^k$. We can define a fibre preserving diffeomorphism $\Phi : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ such that the critical locus of Φ^*S is $M \times \{0\}$. Summing up, any GFQI for the zero section is equivalent to a special generating function whose critical locus is the zero section in $E = M \times \mathbb{R}^k$.

Lemma 2.4.5. Any GFQI of the zero section is equivalent to a special generating function $S : M \times \mathbb{R}^k \to \mathbb{R}$ such that, if $S_x = S(x, \cdot)$,

- S_x coincides with a non degenerate quadratic form q_x at a neighbourhood of $0 \in \mathbb{R}^k$;
- S_x coincides with a non-degenerate quadratic form Q_∞, fixed for all x;
- The only critical point, which is moreover non degenerate, for S_x is $\in \mathbb{R}^k$.

Proof. It is an immediate application of the Generalised Morse Lemma (see [15], pag 149). We only verify that we can in fact apply it: $M \times \{0\}$ really is a critical submanifold of E for S: by definition the vertical differential is 0, and the horizontal vanishes too as S generates the zero section. Moreover, by the transversality requirement on the vertical differential of S, $E_x = \{x\} \times \mathbb{R}^k$ is transverse to $M \times \{0\}$ and x is a non degenerate critical point for $S|_{E_x}$. □

Remark. We remark that due to the absence of critical points other then 0, the signatures of q_x and Q_∞ agree at every point. q_x can in theory change: our efforts will show that we can in fact assume it does not depend on x, and that we can assume S to coincide *everywhere* with q.

Another thing to think about is the following: we made no hypotheses on the signature of Q_∞, hence even showing the point above, we still miss a step: the quadratic forms associated to two generating functions may indeed be different. However, as we did in Section 2.1.1, up to stabilisation and diffeomorphism, we can assume the canonical basis to be orthogonal, where the vectors have length either 1 or -1. The only thing left to fix is the signature: but up to stabilisation, we can always increase the dimensions of the space where the quadratic form is negative definite, and of that where it is positive definite: this finishes the proof. The weirdness one might experience from this (adding an arbitrary, but always finite, number of dimensions to the bundle) comes from the fact that the *stabilisation* does not, as it was defined, induce a proper equivalence relation.

Let now $\varepsilon > 0$ be a small parameter, and let $\mathcal{U}$ be the neighbourhood of $M \times \mathbb{R}^k$ on which S and q agree (see previous lemma). Then we have a clear injection $j : q^{-1}(-\varepsilon) \cap \mathcal{U} \to S^{-1}(-\varepsilon)$. Remark that we may assume, by *stabilisation*, as we already mentioned, that $q^{-1}(-\varepsilon) \neq \emptyset$.

Lemma 2.4.6. If the injection j extends to $j : q^{-1}(-\varepsilon) \to S^{-1}(-\varepsilon)$. Then S and q are equivalent up to a fibre diffeomorphism.

Proof. Extend (the extension of) j to $j : q^{-1}(-\varepsilon) \cup \mathcal{U} \to S^{-1}(-\varepsilon) \cup \mathcal{U}$, which is a diffeomorphism that preserves the fibres. The goal is to make it global on $M \times \mathbb{R}^k$, constructing a further extension J. On $M \times \mathbb{R}^k$ with the Euclidean norm on the fibres, consider the vertical gradient vector fields $X(x, v) = (0, \nabla_v q(x, v))$, $Y(x, v) = (0, \nabla_v S(x, v))$. The flows of X and Y are complete: the first vector field is linear in the coordinates on the whole space, and the second one only outside a compact set which is still enough for the completeness of the flow. The flow hence for all times is a fibre-preserving diffeomorphism of $M \times \mathbb{R}^k$.

Now, if $z \in M \times \mathbb{R}^k$, following its orbit along X, we find at least one $z' \in q^{-1}(-\varepsilon) \cup \mathcal{U}$. Let $Z' = j(z')$, and following the orbit of Z' along Y we find one and only one point Z such that $S(Z) = q(z)$: we have existence and uniqueness of such a point because S and q coincide on $\mathcal{U}$ around the origin, and S is strictly increasing along the gradient lines. The point Z does not depend on the choice of z' because of this strict monotonicity, and because if z', z'' lie in the intersection between a gradient line of X and $q^{-1}(-\varepsilon) \cup \mathcal{U}$, then Z' and Z'' lie in the same Y-orbit. In fact, up to shrinking $\mathcal{U}$ we can assume it to be a ball, and up to reducing ε we can assume the intersection $\mathcal{U} \cap q^{-\varepsilon}$ to be non empty. Now, if $z, z' \in \mathcal{U}$ there is nothing to prove (J on $\mathcal{U}$ acts as the identity, and the two vector fields coincide), and it is impossible for a trajectory to cross $q^{-\varepsilon}$ twice, proving the good definition.

Setting $J(z) = Z$ gives the fibre-preserving diffeomorphism $M \times \mathbb{R}^k \to M \times \mathbb{R}^k$. □

The idea of the following part is to define a fibration, and prove that it has contractible fibres. This will let us consider some smooth global section: because of this we can apply Lemma 2.4.6, and obtain what we wanted.

Let us define the fibration. Up to shrinking the neighbourhood $\mathcal{U}$ in the Lemma, we can suppose it is a product of balls $D^{k-i} \times D^i$ (here D^j denotes the unit ball in $\mathbb{R}^j$, and i is the index of q). Then we can find a diffeomorphism $(\mathcal{U}_x, \mathcal{U}_x \cap S^{-1}(-\varepsilon)) \simeq (D^{k-i} \times D^i, D^{k-i} \times \mathbb{S}^{i-1})$ ($\mathbb{S}^{j-1} = \partial D^j$ is the $j-1$-dimension sphere). The reason for that is simply that S and q coincide on $\mathcal{U}$. Now, since S and Q_∞ coincide outside a compact set K, by Weierstrass Theorem there is a positive $M \in \mathbb{R}$ such that $S^{-1}((-\infty, -M]) \cap K = \emptyset$: therefore with an analogous diffeomorphism one finds that

$$S^{-1}(-\infty) := S^{-1}(-M) \simeq M \times \mathbb{R}^{k-1} \times \mathbb{S}^{i-1}$$

The vertical gradient of S gives us a diffeomorphism between $S^{-1}(\infty)$ and $S^{-1}(-\varepsilon)$: we can therefore identify j with an embedding $q^{-1}(-\varepsilon) \to M \times \mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$. If j_x is the restriction of j to the fibre over x, we can define the fibres of the fibration the following way:

$$\mathcal{P}_x := \left\{ f_x : q_x^{-1}(-\varepsilon) \to \{x\} \times \mathbb{R}^{k-i} \times \mathbb{S}^{i-1} \;\middle|\; f_x \text{ is a diffeomorphism extending } j_x \right\}$$

These sets are considered endowed with the Whitney Weak Topology (see [15]). We have the fibration $\pi : \mathcal{P} = \coprod_{x \in M} \mathcal{P}_x \to M$, $(x, f_x) \to x$. $\mathcal{P}$ is topologised with the disjoint union topology: π is clearly continuous by universal property of the disjoint union topology (it gives a coproduct in **Top**).

Definition 2.4.1 (Fibre bundle)**.** A continuous map $p : E \to B$ is a fibre bundle if for any $y_0 \in Y$ there is a neighbourhood U_{y_0} such that $f^{-1}(U_{y_0})$ is homeomorphic to $U_{y_0} \times f^{-1}(y_0)$. Fibre bundles are also called *locally trivial fibrations.*

We want to prove that π is a locally trivial fibration: to do so, we are going to describe it as the pull-back of a locally trivial fibration. The result will follow thanks to

Lemma 2.4.7. If $p : E \to B$ is a fibre bundle and $f : X \to B$ is continuous, then $\mathrm{pr}_1 : f^*(E) \to X$ is also locally trivial.

Proof. For $x \in X$, one can take a trivialising open neighbourhood U around $f(x)$, and pull it back. The result follows easily noting that $f^{-1}(U)$ is homeomorphic to the graph of $f|_{f^{-1}(U)}$ via (Id, f), and that $p^{-1}(f(x))$ is homeomorphic to $\mathrm{pr}_1^{-1}(x)$. □

Therefore, let us consider Diff, the group of diffeomorphisms of $\mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$, and Emb, the one of the embeddings $D^{k-i} \times \mathbb{S}^{i-1} \to \mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$. The former is endowed with the Whitney Weak Topology and the latter with the Whitney Strong Topology. We omit the proof of the following Lemma: one can find it in [10].

Lemma 2.4.8. The restriction $p :$ Diff$\to$Emb is a fibre bundle.

We can now prove that π is a fibre bundle. The problem is local on M: fix a point $x \in M$. Note that the fibration $q^{-1}(-\varepsilon) \to \mathcal{U}_x$ is a product bundle of fibre $D^{k-i} \times \mathbb{S}^{i-1}$, and that this way we find a continuous map $\Phi : U_x \to$Emb, $y \mapsto j_y$. If we consider Φ^*Diff, it is the set of fibre-preserving diffeomorphisms of $q^{-1}(-\varepsilon)|_{\mathcal{U}_x}$ which extend j: it is exactly $\mathcal{P}|_{\mathcal{U}_x}$ which is then trivial, by Lemmas 2.4.8 and 2.4.7: we achieve our conclusion.

We are going to prove now that π has contractible fibres: this almost gives us the diffeomorphism of Lemma 2.4.6, but we are going to need some arguments more.

Theorem 2.4.9. The fibres of π are contractible for the weak topology when $k > \frac{3}{2}i + 1$.

Proof. Recall the identification of j_x as an embedding $D^{k-i} \times \mathbb{S}^{i-1} \to \mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$. In particular it induces a map of degree 1 between the spheres, just because D^{k-i} and $\mathbb{R}^{k-1}$ are contractible: (any) injection mapping $\mathbb{S}^{i-1}$ to itself diffeomorphically and (the canonical) projection induce isomorphisms in homology. Let us look at the fibration p :Diff$\to$Emb: remember that π is in fact its pull-back, and remark that with the same proof as Lemma 2.4.7, if the fibres of p over a point are contractible, so are those of π on its preimage. We are going to prove then that the fibres over an embedding $\varphi \in$Emb satisfying this degree condition are contractible, finding a homotopy in Emb to a particular class of embeddings.

Let $\bar{g}(u) = (0, u)$ be the standard injection $\mathbb{S}^{i-1} \hookrightarrow \mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$, and $\bar{J}_x$ be the restriction of j_x to $\{0\} \times \mathbb{S}^{i-1}$. As we noted, $\bar{g}$ and $\bar{J}_x$ have the same degree: they are therefore homotopic in Emb$(\mathbb{S}^{i-1}, \mathbb{R}^{k-i} \times \mathbb{S}^{i-1})$ (embeddings from $\mathbb{S}^{i-1}$ into $\mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$) if $k > \frac{3}{2}i + 1$. A proof of this result is in [14]. Remark that however that having the same degree is a sufficient and necessary condition for two classes in $[\mathbb{S}^{i-1}, \mathbb{S}^{i-1}]$ to coincide.

Moreover, the restriction map Emb$\to$Emb$(\mathbb{S}^{i-1}, \mathbb{R}^{k-i} \times \mathbb{S}^{i-1})$, $f \mapsto f|_{\{0\} \times \mathbb{S}^{i-1}}$ is also a fibre bundle (see [10]): it satisfies to the path lifting property, and we have a path in Emb connecting j_x (above the fibre of $\bar{J}_x$) to an embedding g extending $\bar{g}$.

Now, write $g(z, u) = (a(z, u), b(z, u))$, and consider the path $g_t(z, u) = (t^{-1}a(tz, u), b(tz, u))$ for $t \in (0, 1]$. It is clearly a continuous path in Emb, and it converges to $h(z, u) = (d_0^{D^{i-1}} a(\cdot, u).z, u) \in$Emb for $t \to 0$. It is easy to show that h actually is an embedding. If we prove then that the fibre $p^{-1}(h)$ is contractible, we are done: in fact we have a path in Emb connecting h and j_x, therefore all the fibres are homeomorphic, $p^{-1}(j_x) = \mathcal{P}_x$. But we can use a similar trick again: if $H \in$Diff is a lift of h, it is linear around the origin. Write $H_t(t^{-1}a'(tz, u), b'(tz, u))$ for $t \in (0, 1]$. $H_1 = H$, and by linearity in $D^{k-1} \times \mathbb{S}^{i-1}$, $\lim_{t\to 0} H_t(z, u)$ is the trivial extension of h, defined through the same formula but on the whole space $\mathbb{R}^{k-i} \times \mathbb{S}^{i-1}$. □

The fibres of $\pi : \mathcal{P} \to M$ being contractible, there is a global section for this fibration. However, we are not sure about its differentiability, and even about the definition of such differentiability; the problem lies in the fact that, even if we are sure about the differentiability on the fibres, we do not know, a priori, whether the section would also be differentiable changing the base point. For further discussion, we refer to [30], [10] and [32]. What matters towards our discussion is that the section is indeed differentiable, and that we have the following Corollary:

Corollary 2.4.10. The fibration $\pi : \mathcal{P} \to M$ has a global section: $q^{-1}(-\varepsilon)$ and $S^{-1}(-\varepsilon)$ are diffeomorphic as vector bundles over M, via a diffeomorphism extending $j : q^{-1}(-\varepsilon) \cap \mathcal{U} \to S^{-1}(-\varepsilon)$.

We have thus finished the proof of Viterbo's Uniqueness Theorem.

2.5 The non-exact case

We are going to reduce the study of the non-exact case to the discussion we already made. We are going to make two similar steps: how to define an exact Lagrangian submanifold out of a non-exact one, and how to define a Hamiltonian isotopy out of a symplectic isotopy.

Let $L \leq T^*M$ be a Lagrangian submanifold, $p : T^*M \to M$ the canonical projection, $e : L \to T^*M$ the embedding, and $p_L = p \circ e$ the projection of L on M. We need to make the assumption that p_L and e induce isomorphisms $p_L^* : H^1(M) \to H^1(L)$ and $e^* : H^1(T^*M) \to H^1(L)$. Thanks to this hypothesis, we can take a closed form $\bar{\alpha} \in Z^1(M)$ such that $p_L^*(\bar{\alpha}) = i_L^*(\lambda)$ (λ is the tautological 1-form on T^*M). We define the isotopy $\mu_t : T^*M \to T^*M$ via $\mu_t(x, y) = (x, t - t\bar{\alpha}(x))$. It is symplectic, since an easy calculation shows $\mu_t^*\lambda = \lambda - tp^*\bar{\alpha}$. Furthermore, $\mu_1(L)$ is exact Lagrangian: if $e_1 = \mu_1 \circ e$, we have

$$[e_1^*\lambda] = [e^*\mu_1^*\lambda] = [e^*(\lambda - \bar{\alpha})] = [e^*\lambda - e^*p^*\bar{\alpha}] = [e^*\lambda - p_L^*\bar{\alpha}] = 0$$

We want to find a generating function for $\mu_1(L)$. Suppose that $\alpha \in Z^1(E)$, for some vector bundle $\pi : E \to M$. Then $\pi^*(\bar{\alpha})$ and α are cohomologous: since π^* and p_L induce isomorphisms in cohomology, we actually need to show that

$$[p_L^*\bar{\alpha}] = [e^*p^*(\pi^*)^{-1}\alpha]$$

which is true by the hypothesis we made on $\bar{\alpha}$ and the fact that the one on the right is the expression for $e^*\lambda$ (see the proof of Lemma 2.1.2). If then $S \in \mathcal{C}^\infty(E)$, $dS = \pi^*\bar{\alpha} - \alpha$, S is a generating function for $\mu_1(L)$ (we also find again the exactness of $\mu_1(L)$).

Let us now consider a symplectic isotopy $(\varphi_t)_{t\in I}$ of T^*M. We can then choose a smooth family of forms on M, $(\beta_t) \subset Z^1(M)$, such that $[p^*\beta_t] = [\varphi_t^*\lambda - \lambda]$ (again, because p induces an isomorphism in cohomology). With similar procedure and calculations as above, we define the symplectic isotopy $\eta_t(x, y) = (x, y - \beta_t(x))$: then $(\eta_t \circ \varphi_t)^*\lambda$ is then exact at all times t, so that $(\eta_t \circ \varphi_t)_{t\in I}$ is a Hamiltonian isotopy.

We can now prove Sikorav' Existence Theorem (Theorem 2.1.5) in the non-exact setting. The result will be weaker: the generating form will only be closed, not necessarily exact.

Proof. We consider the symplectic isotopy $(\varphi_t)_{t\in I}$. Then $(\varphi_t \circ \mu_1^{-1})_{t\in I}$ is also a symplectic isotopy, which we can compose with the (η_t) we constructed before: $(\eta_t \circ \varphi_t \circ \mu_1^{-1})_{t\in I}$ is Hamiltonian. We know that $\mu_1(L)$ has a generating function, by Theorem 2.1.5 (exact case). Therefore by the same Theorem also $\varphi_1 \circ \eta_1(L)$ has one, $S_1 : E \to \mathbb{R}$. Using the same approach as before we can find a closed form $\gamma \in Z^1(M)$ such that $dS_1 - \pi^*\gamma$ is a generating form for $\varphi_1(L)$: remark in fact that, noting $e_{\varphi_1(L)} = \varphi_1 \circ e$ the embedding of $\varphi_1(L)$ and $p_{\varphi_1(L)}$ the respective projection on M, $p_{\varphi_1(L)}$ also induces an isomorphism in cohomology. □

And now, we can prove Viterbo's Uniqueness Theorem in the non-exact case.

Proof. We prove the persistence of the uniqueness property under symplectic isotopies: the zero section is exact, and we already know the uniqueness of the GFQI for it.

Let $(\varphi_t)_{t\in I}$ be a symplectic isotopy, $e : L = L_0 \to T^*M$ an embedded Lagrangian submanifold having the uniqueness property, and denote $L_t = \varphi_t(L_0)$ the corresponding path of Lagrangian submanifolds, with the embeddings $e_t = \varphi_t \circ e$ and the projections $p_t = p \circ \varphi_t \circ e$. We use the usual argument, and find a smooth family in $Z^1(M)$, $\bar{\alpha}_t$, such that $[e_t^*\lambda - \lambda] = [p_t^*\bar{\alpha}_t]$ for all times. We can deform the path as we did at the beginning of this section: let us define the symplectic isotopy $\bar{\eta}_t(x, y) = (x, y - \bar{\alpha}_t(y))$ and the path of exact Lagrangian submanifolds $\tilde{L}_t = (\bar{\eta}_t \circ \varphi_t)L_0$. $\tilde{L}_0 = L_0$ has the uniqueness property, so $\tilde{L}_1$ does by Viterbo's uniqueness Theorem in the exact case, and we finish with the same argument as in the last proof. $\square$

Chapter 3

Floer and Morse Theory on Generating Functions

This chapter will be devoted to the proof of an isomorphism between Floer theory of the action functional and Morse homology of the generating functions, following [25].

3.1 The Action Functional as a generating function

We generalise here the definitions we gave above, to speak of generating functions defined on fibrations in general. Let $p : E \to M$ be a submersion: then the notion of generating function $S \in \mathcal{C}^\infty(E)$ is well defined, as follows. Since p is a submersion (we do not technically require it to be surjective), for every $x \in M$, $p^{-1}(x)$ is a submanifold of E, so the differential of S restricted to such submanifolds is well defined. We therefore set, if for $x \in M$ $E_x = p^{-1}(x)$,

$$\Sigma_S := \left\{ e \in E \;\middle|\; p(e) = x,\, d_e^{E_x} S = 0 \right\}$$

which is a generalisation of the notation $\partial^\lambda S(x, \lambda) = 0$ we adopted earlier on. We still need the transversality property to ensure that Σ_S is still a submanifold of E. Now, since dp is surjective, there is a right section r to it (at least locally). Let us therefore define the partial derivative parallel to the manifold M the following way: $\partial S(e) = d_e S \circ r$. This way, $\partial S(e) \in T^*_{p(e)} M$. We recover the definition of L_S in a more abstract language:

$$L_S = \{ (x, \partial S(e)) \mid p(e) = x,\, e \in \Sigma_S \}$$

We mention that Serre fibrations $p : X \to M$ are clearly surjective submersions: for the surjectivity, if $x \notin p(X)$ then by connectedness of M we can pick any curve connecting a point in the image of p to x and lift it, finding the contradiction. The fact that p is a submersion is proved exactly the same way, lifting a curve representing a tangent vector.

We are now going to prove, formally[1], that the action functional is a generating function. Let H be a (time-dependent) Hamiltonian function on T^*M, and define the action functional

$$\mathcal{A}_H(\gamma) := \int_\gamma pdq - \int_0^1 H_t(\gamma(t))dt \tag{3.1}$$

on the space of paths with 0 initial momentum

$$\Omega := \{ \gamma : I \to M \mid \gamma(0) \in 0_M \}$$

We have a fibration $p : \Omega \to M$ given by $p(\gamma) = \gamma(1)$. With some calculations, given that a tangent vector ξ at $\gamma \in \Omega$ are represented by deformations $\eta : (-\varepsilon, \varepsilon) \times I \to M$ such that $\eta(0,t) = \gamma(t)$, $\eta(s,0) \in 0_M$, $\partial s|_{s=0}\eta(s,t) = \xi(t)$, we have the following expression for the differential of $\mathcal{A}_H$:

$$d_\gamma \mathcal{A}_H.\xi = \int_0^1 \omega(\dot{\gamma} - X_H(\gamma), \xi)dt + \langle \lambda\gamma(1), \xi(1)\rangle - \langle \lambda\gamma(0), \xi(0)\rangle \tag{3.2}$$

Here, $\omega = -d\lambda$ is the canonical symplectic form on T^*M. Since $\gamma(0) \in 0_M$, the last term is 0. Now, let us compute the restriction of the differential to the fibres. Since the endpoint of the curves need to be constant, $\xi(1) = 0$ for every vector which is tangent to $p^{-1}(x)$, for a fixed $x \in M$. Setting the restriction of the differential to be zero, we recover the condition

$$\dot{\gamma} - X_H(\gamma) = 0 \text{ i.e. } \gamma(t) = \phi_H^t(x)$$

We have thus found $\Sigma_{\mathcal{A}_H}$. Let us compute $\partial\mathcal{A}_H$: if $x \in M$, $v \in T_xM$ and γ is a curve such that $\gamma(1) = \phi_H^1(x') = x$ for some $x' \in 0_M$, we have by construction

$$\partial\mathcal{A}_H(\gamma)(v) = d_{\phi^{\cdot}_H(x')}\mathcal{A}_H.(dr.v) = \langle \lambda\phi_H^1(x'), (dr.v)(1)\rangle$$

By definition $dr.v(1) = v$, hence $\partial\alpha_H(\gamma) = \phi_H^1(x')$. This proves that the action functional is a generating function for the submanifold $L_{\mathcal{A}_H} = \phi_H^1(0_M)$.

3.2 Morse homology of generating functions

Let us consider the cotangent bundle (T^*M, ω) with its standard symplectic structure. Given a Lagrangian submanifold $L \leq T^*M$, we can represent it by the results in [8] via a GFQI $S : E = M \times \mathbb{R}^k \to \mathbb{R}$. We showed as in [30] that the attribution $S \mapsto L_S$ is a Serre fibration. We denote as $\mathcal{S}_{Q:E}$ the set of generating functions (hence smooth functions satisfying the usual

[1]We shall not discuss Banach manifold structures on path spaces.

transversality condition) on E which coincide with the non-degenerate quadratic form Q outside a compact subset.

What we remark immediately is that intersections between L and 0_M correspond to the critical points of the differential dS for obvious reasons, so in principle one could study them using Morse Theory. By replacing S with a good approximation, we can assume the critical points to be non degenerate.

For $i : N \hookrightarrow M$ a closed manifold, we define the relative Morse chain complex as the free group generated by critical points of $S|_{E_N}$, where E_N is the restriction of E to N. Let us denote the free abelian group generated by set of critical points of $S|_{E_N}$ of index p as $CM_p(S, N : E)$. We can then define the Morse boundary operator with the following procedure: let us consider the Morse vector field

$$\dot{x} = -\nabla_g S(x)$$

where g is a Riemannian metric on $E|_N$, a priori without any constraints. Define the energy of a solution γ as $E(\gamma) = \int_{\mathbb{R}} \|\dot{\gamma}(t)\|_g^2 dt$. One can prove that solutions are indeed defined for all real times, that under the generic choice of the metric they connect two critical points of S and that in this case the set

$$\mathcal{M}_{g,S,N}(x^-, x^+) = \left\{ \gamma \in \mathcal{M}_{g,S}(M, N) \;\middle|\; \lim_{t \to \pm\infty} \gamma(t) = x^\pm \right\}$$

is a smooth manifold, of dimension the difference of the indices of x^+ and x^-. In the definition, $\mathcal{M}_{g,S}(M, N)$ denotes the set of integral curves $\gamma : \mathbb{R} \to E|_N$ of the Morse vector field, with finite energy. $\mathbb{R}$ acts on $\mathcal{M}_{g,S,N}(x^-, x^+)$ via translation on the time, and the quotient is an oriented compact manifold of dimension $n_S(x^+) - n_S(x^-) - 1$: in particular if $n_S(x) - n_S(y) = 1$, the corresponding moduli space is a set of finite points, each of this with a sign. We set $n(x, y)$ as the sum of the signs of these points, and define the boundary operator as

$$CM_p(S, N : E) \ni x \mapsto \partial x = \sum_{y \in C_{p-1}(S,N:E)} n(x, y)y$$

One can check that $\partial^2 = 0$, and we obtain this way the relative homology groups $HM_\bullet(S, N : E)$.

We quickly remark that since S is quadratic at infinity, all its critical points lie in a compact subset of E, therefore the results valid in the compact manifold case still hold.

One can also introduce a filtration on the chain complex: if $\lambda \in \mathbb{R}$, we can define $C_p^\lambda(S, N : E)$ with the further requirement that the value of S at the generators be less than or equal to λ. The differential ∂ respects this filtration, that is

$$\partial CM_p^\lambda(S, N : E) \le CM_{p-1}^\lambda(S, N : E)$$

The reason is simply that the value of S decreases along the gradient trajectories. We thus have a filtration in the homology groups $H^\lambda(S, N : E)$. The inclusion $j_\#^\lambda : CM_p^\lambda(S, N : E) \to CM_p(S, N : E)$ induces a morphism at the homology level,

$$j_*^\lambda : HM_\bullet^\lambda(S, N : E) \to HM_\bullet(S, N : E)$$

3.3 Floer theory and the Action Functional

As we pointed out above, in the first section, the action functional is a very natural generating function. What one can do is study its Floer homology. We are going to present it here, briefly, in its relative version.

We use the definition (3.1) whose differential was already computed in (3.2). As above, we restrict to Ω, but we impose the further condition that the endpoint of the paths lie in the conormal bundle of a closed submanifold N of M:

$$\Omega(N) = \{ \gamma \in \Omega \mid \gamma(1) \in \nu^* N \}$$

Under such conditions, the differential simplifies and becomes

$$d_\gamma \mathcal{A}_H.\xi = \int_0^1 \omega(\dot\gamma - X_H(\gamma), \xi) dt$$

To define the gradient flow for the action, we need a metric. If $(J_t)_t \in I$ is a path in the space of almost-complex structures in T^*M calibrated by ω, we have the L^2 metric on $T\Omega(N)$

$$\langle \xi | \eta \rangle_{L^2} = \int_0^1 \omega(\xi(t), J_t \eta(t)) dt$$

Clearly, we identified the tangent vectors to a curve $\gamma \in \Omega(N)$ as vector fields on its support, which are sections of the pullback bundle $\gamma^* T(T^*M) \to I$.

Since ω is J_t-invariant and $J_t^2 = -1$, an integral curve for (minus) the gradient flow of $d\mathcal{A}_H$ $u : \mathbb{R} \times I \to T^*M$ satisfies to the conditions:

$$\begin{cases} \partial_s u + J_t \left(\partial_t u - X_H(u) \right) = 0 \\ u(\cdot, 0) \in 0_M, \;\; u(\cdot, 1) \in \nu^* N \end{cases} \tag{3.3}$$

If we furthermore require that the action functional be bounded on a trajectory, there are two integral curves γ^+ and γ^- for X_H, with endpoints in 0_M and $\nu^* N$, such that

$$\lim_{s \to \pm\infty} u(s,t) = \gamma^\pm(t)$$

A Maslov index for such a path can be defined, and shall provide the grading for the complex. We define the p-th Floer chain group as the free abelian group generated by the set of these paths with Maslov index p $CF_p(H, N : M)$. The set of solutions of (3.3) with bounded action will be denoted $\mathcal{M}(J, H, N : M)$. As we did in Morse theory, $\mathbb{R}$ acts via translation on the variable s on the set

$$\mathcal{M}_{(J,H)}(\gamma^-, \gamma^+) := \left\{ u \in \mathcal{M}(J, H, N : M) \;\middle|\; \lim_{s \to \pm\infty} u(s, \cdot) = \gamma^\pm \right\}$$

and when the difference $m(\gamma^+) - m(\gamma^-) = 1$, the quotient is an oriented, compact 0 dimensional manifold. We define the differential of the Floer complex formally in the same way as we did for the Morse complex. Analogous reasons, albeit more technically difficult to prove, show that the differential squares to 0: we can define the Floer relative homology groups: $HF_\bullet(J, H, N : M)$.

The action functional also gives us a filtration, which is the infinite-dimensional analogue of the one we defined in the previous section. The boundary homomorphism still respects it, as the action functional decreases along the gradient lines.

3.4 Interpolation and proof of the main result

Turns out that the two points of view are equivalent. More precisely, we are going to prove the following result:

Theorem 3.4.1. Let $L = \phi_H^1(0_M)$, with a GFQI S. Then, for any $\lambda \notin \mathrm{Spec}(L : N)$, there is a level-preserving isomorphism

$$\begin{array}{ccc} H_\bullet^\lambda(S, N : E) & \xrightarrow{\cong} & HF_\bullet^\lambda(J, H, N : M) \\ \downarrow j_*^\lambda & & \downarrow j_*^\lambda \\ H_\bullet(S, N : E) & \xrightarrow{\cong} & HF_\bullet(J, H, N : M) \end{array}$$

The result is valid in cohomology as well.

We have not yet defined the set $\mathrm{Spec}(L : N)$ which appears in the statement of the Theorem. It is called "**action spectrum**", and is nothing but the set of critical values of the action functional (in this case the setting will be a bit different from the usual one, the proof shall make clear why and how $\mathrm{Spec}(L : N)$ is the set of critical values of S_N).

To prove the Theorem 3.4.1 we are going to interpolate the two approaches: basically, we add a term to the classic action functional, in order to keep the informations provided by the generating function. We are then going to consider variations in the Hamiltonian H and the generating function, to prove that under (regular) homotopies we find isomorphic homology theories. We shall then find two good endpoints for such a homotopy, and this will more or less prove the Theorem.

3.4.1 Spectrum of the Action Functional

Let $E = M \times \mathbb{R}^m$. Clearly, $T^*E = T^*M \oplus \mathbb{C}^m$; let H be a compactly supported Hamiltonian on T^*E. Let us also fix a generating function $S \in \mathcal{S}_{(Q:E)}$, and define the space of paths $\mathcal{P}(S : E)$ in T^*E with the first endpoint in $\mathrm{Graph}(dS)$. Computing the differential of the action functional as in Section 3.1, we find the expression for $\Gamma \in \mathcal{P}(S : E)$ and $\eta \in T_\Gamma \mathcal{P}(S : E)$

$$d_\Gamma \mathcal{A}_H.\eta = \int_0^1 \Big\{ \omega \oplus \omega_0(\dot{\Gamma}, \eta) - d_\Gamma H.\eta \Big\} + \langle \Gamma(1), d_{\Gamma(1)}\pi.\eta(1)\rangle - d_{\pi(\Gamma(0))}S.d_{\Gamma(0)}\pi.\eta(0) \tag{3.4}$$

ω and ω_0 are the standard symplectic forms of T^*M and $\mathbb{C}^m$. Setting then

$$\mathcal{A}_{(H,S)}(\Gamma) = \mathcal{A}_H(\Gamma) + S(\pi(\Gamma(0))) \tag{3.5}$$

by the identity (3.4),

$$d_\Gamma \mathcal{A}_{(H,S)}.\eta = \int_0^1 \left\{ \omega \oplus \omega_0(\dot{\Gamma}, \eta) - d_\Gamma H.\eta \right\} dt + \langle \Gamma(1), d_{\Gamma(1)}\pi.\eta(1) \rangle \tag{3.6}$$

This shows, via the same proof as in Section 3.1, that if H is identically 0 on $\mathbb{C}^m$, $\mathcal{A}_{(H,S)}$ is a generating function for the submanifold $\phi_H^1(L_S)$, using the fibration

$$p : \mathcal{P}(S : E) \to M \quad p(\Gamma) = \pi_E \circ \pi_{T^*E}(\Gamma(1))$$

Once again, we define the relative version $\mathcal{P}(S, N : E)$ for a closed submanifold of M taking paths with the second endpoint in the conormal bundle $\nu^*E|_N$:

$$\mathcal{P}(S, N : E) = \{ \Gamma : I \to T^*E \mid \Gamma(0) \in \mathrm{Graph}(dS), \Gamma(1) \in \nu^*N \times 0_M \} \tag{3.7}$$

Restricting the action functional, only the integral term in (3.6) survives. The critical points of $\mathcal{A}_{(H,S)}$ are therefore paths in T^*E satisfying Hamilton's equations, starting from $\mathrm{Graph}(dS)$ and ending in $\nu^*E|_N$.

We have now all the necessary notions to define the spectrum of the action properly. Let $\mathrm{Crit}(H, S, N : E)$ be the set of critical points of the action functional $\mathcal{A}_{(H,S)}$ restricted to $\mathcal{P}(S, N : E)$. The (relative) action spectrum will therefore be defined as

$$\mathrm{Spec}(H, S, N : E) = \mathcal{A}_{(H,S)}\mathrm{Crit}(H, S, N : E)$$

Given that the isomorphism of Theorem 3.4.1 only exists for $\lambda \notin \mathrm{Spec}(L : N)$, there are some considerations to make. First, the action spectrum does not depend on the choice of (H, S), but only on the Lagrangian submanifold $\phi_H^1(L_S)$ (see: [27]); furthermore for $N = \emptyset$ we find the "absolute" action spectrum. Secondly, we need to know how small the action spectrum is: the following Lemma answers to this question.

Lemma 3.4.2. $\mathrm{Spec}(H, S, N : E)$ is a compact and nowhere dense set of $\mathbb{R}$.

Proof. Since $\mathrm{Spec}(H, S, N : E)$ is clearly closed in $\mathrm{Spec}(H, S, E)$, it suffices to show that the latter is a bounded subset of $\mathbb{R}$; we are going to do so applying Ascoli-Arzelà's Theorem. Let us fix an almost complex structure calibrated by $\omega \oplus \omega_0$, J, on T^*E, so that $(T^*E, \omega \oplus \omega_0(\cdot, J\cdot))$ is a metric space. Since I is compact, to use the Theorem we need to prove the equicontinuity of the family

$$\mathcal{F} = \left\{ \Gamma : I \to T^*E \;\middle|\; \Gamma(0) \in 0_M, \dot{\Gamma} = X_H \right\} \tag{3.8}$$

and that for every t in the unit interval the set $\mathcal{F}(t) = \{ \Gamma(t) \mid \Gamma \in \mathcal{F} \}$ is relatively compact in T^*E.

H is compactly supported, hence there is an $N \in \mathbb{R}_{>0}$ such that $\|X_H\|_g \leq N$. To prove the equicontinuity, pick $s \leq t \in I$. Then for a $\Gamma \in \mathcal{F}$ there is a partition $s = t_0 \leq t_1 \leq \cdots \leq t_n = t$ such that $\Gamma([t_i, t_{i+1}])$ is in a trivialising open set. Then:

$$d_g(\Gamma(s), \Gamma(t)) \leq \sum_{i=0}^{n-1} d_g(\Gamma(t_i), \Gamma(t_{i+1})) \leq M(t-s)$$

where we used the definition of $\mathcal{F}$. Remark that the bound does not depend on the partition nor on the chosen curve: we have thus proved the equicontinuity.

To prove that $\mathcal{F}(t)$ is relatively compact, take a sequence $(\Gamma_n) \subset \mathcal{F}$. Then, since H is compactly supported, its integral curves need to stay in a compact subset of T^*E, K (as $X_H = 0$ outside of K). This implies that up to taking a subsequence, the $\Gamma_n(t)$ need to converge to a point $y \in T^*E$, which shows that $\mathcal{F}(t)$ is relatively compact.

By Ascoli-Arzelà, $\mathcal{F}$ is relatively compact; in general however $\mathcal{A}_{(H,S)}$ is not continuous for the $\mathcal{C}^0$ topology, since the temporal derivatives of the curves appear in the definition. In this case nevertheless the temporal derivative of the curves equal X_H, for a smooth Hamiltonian: this implies easily that $\mathcal{A}_{(H,S)}$ is continuous on $\mathcal{F}$ with the $\mathcal{C}^0$ topology. With some effort, one could prove that the family $\mathcal{F}$ is also closed for the $\mathcal{C}^0$ topology[2]: $\mathcal{A}_{(H,S)}(\mathcal{F})$ is the continuous image of a compact set, and therefore is a compact, hence bounded, subset of $\mathbb{R}$. This gives us the compactness of the spectre.

To prove that the spectre is nowhere dense, we define the function

$$f : \nu^*(N \times \mathbb{R}^m) \to \mathbb{R}, \quad f(x) = \mathcal{A}_{(H,S)}(\gamma_x)$$

where $\gamma_x(t) = \phi_H^t \circ \left(\phi_H^1\right)^{-1} x$. Then, by chain rule, the set $\mathrm{Spec}(H, S, N : E)$ is contained in the set of critical values of f. f being smooth, the latter is nowhere dense by Sard's Theorem. □

3.4.2 The interpolation

To interpolate, as we said above, fix two "target" generating functions S_α, S_β, two target Hamiltonians H_α, H_β and two target almost complex structures J_α, J_β. Let us consider a homotopy $(S^{\alpha,\beta}, H^{\alpha,\beta}, J^{\alpha,\beta})$ such that, for a positive R large enough,

$$(S_s^{\alpha,\beta}, H_s^{\alpha,\beta}, J_s^{\alpha,\beta}) = \begin{cases} S^\alpha, H^\alpha, J^\alpha & \text{for } s \leq -R \\ S^\beta, H^\beta, J^\beta & \text{for } s \geq R \end{cases}$$

[2]What is harder to prove is the convergence of the first derivatives. One can achieve it using that, for a sequence of functions defined on the interval I that converges at least on one point, and whose derivatives converge uniformly, is uniformly convergent and one can swap limit and derivative.

We can proceed the same way we did in the case where S, H and J were fixed: we can define the space $\mathcal{M}_{(J^{\alpha,\beta},H^{\alpha,\beta},S^{\alpha,\beta})}(N:E)$ of the solutions of

$$\begin{cases} \bar{\partial}_{J^{\alpha,\beta},H^{\alpha,\beta}}u := \partial_s u + J^{\alpha,\beta}\left(\partial_t u - X_{H^{\alpha,\beta}}(u)\right) = 0 \\ u(s,0) \in \mathrm{Graph}(dS_s^{\alpha,\beta}), \quad u(s,1) \in \nu^*(N) \times 0_{\mathbb{R}^m} \end{cases} \tag{3.9}$$

with finite energy

$$E(u) = \int_{\mathbb{R}} ds \int_0^1 dt \left\{ \|\partial_s u\|^2_{J^{\alpha,\beta}} + \|\partial_t u - X_{H^{\alpha,\beta}}(u)\|^2_{J^{\alpha,\beta}} \right\} < \infty \tag{3.10}$$

We remark that clearly the norm induced by $\omega(\cdot, J_s^{\alpha,\beta}\cdot)$ depends on s.By standard regularity theory means, they are smooth. If we define a L^2 metric on $\mathcal{P}(S^\alpha, N : E)$ and $\mathcal{P}(S^\beta, N : E)$ (using the almost complex structures J^α and J^β), we can see that for times $|s| > R$ the solutions coincide with the standard gradient lines. We can give a similar definition for the set $\mathcal{M}_{(J^i,S^i,H^i)}(N:E)$, $i \in \{\alpha, \beta\}$. We shall omit the i when talking about a generic set of fixed parameters. Remark that there is an important difference between the two cases: when the parameters (J, H, S) do not depend on the parameter s (J, H in general depend on the time t), the solutions of (3.9) are really the lines $-\mathrm{grad}\mathcal{A}_{(H,S)}$ for the L^2 metric defined by J; this is not true during the deformation of the parameters since there's not even an L^2 metric defined.

We also impose the conditions for infinite time: asking that for all $t \in I$,

$$\lim_{s\to-\infty} u(s,t) = x^\alpha(t) \tag{3.11}$$

$$\lim_{t\to\infty} u(s,t) = x^\beta(t) \tag{3.12}$$

where x^α, x^β are two paths, respectively in $\mathcal{P}(S^\alpha, N : E)$ and $\mathcal{P}(S^\beta, N : E)$, satisfying Hamilton's equations respectively for H^α and H^β. The set of the solutions of (3.9) satisfying (3.11) will be denoted $\mathcal{M}_{(J^{\alpha,\beta},H^{\alpha,\beta},S^{\alpha,\beta})}(x^\alpha, x^\beta)$. We similarly define $\mathcal{M}_{(J^i,H^i,S^i)}(x^i, y^i)$ for $i \in \{\alpha, \beta\}$.

We would like to define a complex the same way we did in Section 3.3; one could take regular homotopies such that $\mathcal{M}_{(J^{\alpha,\beta},H^{\alpha,\beta},S^{\alpha,\beta})}(x^\alpha, x^\beta)$ are manifolds, and for the orientation we have the following lemma:

Lemma 3.4.3. For regular homotopies $(J^{\alpha,\beta}, H^{\alpha,\beta}, S^{\alpha,\beta})$ and for any critical points x^α, x^β, the determinant bundle

$$\mathbf{Det} \to \mathcal{M}_{(J^{\alpha,\beta},H^{\alpha,\beta},S^{\alpha,\beta})}(x^\alpha, x^\beta)$$

is trivial: $\mathcal{M}_{(J^{\alpha,\beta},H^{\alpha,\beta},S^{\alpha,\beta})}(x^\alpha, x^\beta)$ is thus oriented. Moreover, we can suppose the orientations to be coherent.

Proof. See [27]. □

We remark that the Lemma holds in the case of fixed, but regular, triplets (J, H, S).

We are going to state a Theorem, whose content is as follows: on one side, we can still define a Floer complex taking curves with first endpoint in Graph(dS) instead of 0_M, and on the other side the Floer homology groups do not depend on the choices of almost complex structures, Hamiltonians and generating function.

First, we define the Floer chain complex: $CF_p(H, S, N : E)$ is the free abelian group generated by the critical points of $\mathcal{A}_{(H,S)}$ restricted to $\mathcal{P}(S, N : E)$ of index p (grading defined in as in the case without second endpoint in $\nu^* N \times 0_M$). Given the orientations in Lemma 3.4.3, one can define the numbers $n(x, y)$ for $x \in CF_p(H, S, N : E)$, $CF_{p-1}(H, S, N : E)$ as the algebraic number of points in the 0-dimensional compact manifold $\mathcal{M}_{(J,H,S)}(N : E)/\mathbb{R}$.

Theorem 3.4.4. For a regular triplet (J, H, S) one can define boundary morphisms $\partial_p : CF_p(H, S, N : E) \to CF_{p-1}(H, S, N : E)$ as

$$x \mapsto \partial x = \sum_{y \in CF_{p-1}(H,S,N:E)} n(x, y)y$$

We define the Floer Homology groups

$$HF_p(J, H, S, N : E) = \frac{\ker \partial_p}{\operatorname{Im} \partial_{p+1}}$$

Fix now regular parameters $(J^\alpha, H^\alpha, S^\alpha)$, $(J^\beta, H^\beta, S^\beta)$. There are canonical isomorphisms

$$H_\bullet(J^\alpha, H^\alpha, S^\alpha, N : E) \to HF_\bullet(J^\beta, H^\beta, S^\beta, N : E)$$

such that $h_{\alpha\alpha} = Id$, $h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma}$

We can still define the following filtration at the chain level:

$$Crit_p^\lambda(H, S, N : E) := Crit_p(H, S, N : E) \cap \mathcal{A}^{-1}_{(H,S)}((-\infty, \lambda))$$

Since the action decreases along the elements of $\mathcal{M}_{(J,H,S)}(N : E)$, the boundary morphism respects the filtration: their restrictions will be noted ∂^λ, and they give a filtration in the homology

$$HF_p^\lambda(J, H, S, N : E) := \frac{\ker \partial_p^\lambda}{\operatorname{Im} \partial_{p+1}^\lambda}$$

We still have morphisms induced by the injections:

$$j_*^\lambda : HF_\bullet^\lambda(J, H, S, N : E) \to HF_\bullet(J, H, S, N : E)$$

The hope would be that the isomorphisms induced by homotopies in Theorem 3.4.4 respect the filtration. This is clearly not true, since changes in the Hamiltonian and the generating function modify the values of the action functional, but we can estimate the variation:

Theorem 3.4.5. Fix two sets of regular parameters (H^α, S^α), (H^β, S^β). Then if

$$\varepsilon^{\alpha,\beta}_{(H,S)} := -\int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha)$$

we have the following commutative diagram:

$$\begin{array}{ccc} HF^{\lambda+\varepsilon^{\alpha,\beta}_{(H,S)}}_\bullet(J, H^\beta, S^\beta, N : E) & \xrightarrow{j_*^{\lambda+\varepsilon^{\alpha,\beta}_{(H,S)}}} & HF_\bullet(J, H^\beta, S^\beta, N : E) \\ h_{\alpha,\beta}\uparrow & & h_{\alpha,\beta}\uparrow \\ HF^\lambda_\bullet(J, H^\alpha, S^\alpha, N : E) & \xrightarrow{j_*^\lambda} & HF_\bullet(J, H^\alpha, S^\alpha, N : E) \end{array}$$

Proof. The commutativity is obvious since the j_* are induced by inclusions. The aim of the proof is to estimate the difference $\mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha) - \mathcal{A}_{(H^\beta,S^\beta)}(x^\beta)$, where x^α, x^β is a pair of critical points for the respective action functionals. To do so, we need to define a homotopy between (H^α, S^α) and (H^β, S^β). Since the morphisms $h_{\alpha,\beta}$ are defined through regular homotopies (homotopies who make the space of solutions a smooth manifold), we need this homotopy to be regular. Now, fix a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(s) = 0$ for $s \in \mathbb{R}_{\le 0}$ and $\rho(s) = 1$ for $s \in \mathbb{R}_{\ge 1}$. Clearly the homotopy

$$(\rho(s)H^\beta + (1 - \rho(s))H^\alpha, \rho(s)S^\beta + (1 + \rho(s))S^\alpha)$$

has no reason to be regular; however, we can approximate it in the $\mathcal{C}^1$-topology using via a regular homotopy (H_s, S_s) (usual transversality argument).

We want to apply the Fundamental Theorem of Calculus: we compute

$$\frac{d}{ds}\mathcal{A}_{(H_s,S_s)}(u(s,\cdot)) = d_{u(s,\cdot)}\mathcal{A}_{(H_s,S_s)}.\partial_s u(s,\cdot) - \int_0^1 \partial_s H_s(u(s,t))dt + \partial_s S_s(u(s,0))$$

for a line u connecting x^α and x^β, satisfying (3.9). Now, there is an arbitrarily small positive constant ε such that

$$|\partial_s H_s(x) - \rho'(s)(H^\beta - H^\alpha)(x)| < \varepsilon$$
$$|\partial_s S(x) - \rho'(s)(S^\beta - S^\alpha)(x)| < \varepsilon$$

given the approximation property of (H_s, S_s) with respect to the homotopy defined via ρ. In the following calculation we are going to integrate on $\mathbb{R}$. We remark however that the image of u in T^*E is clearly compact, which allows us to assume the integral of the function constantly equal to ε over this curve to be finite and in fact of order ε. ε. Therefore

$$\mathcal{A}_{(H^\alpha,S^\alpha)}(x^\alpha) - \mathcal{A}_{(H^\beta,S^\beta)}(x^\beta) = \int_\mathbb{R} \frac{d}{ds}\mathcal{A}_{(H_s,S_s)}(u(s,\cdot))ds \le$$

$$\le \varepsilon + \int_\mathbb{R} ds \left\{ d_{u(s,\cdot)}\mathcal{A}_{(H_s,S_s)}.\partial_s u(s,\cdot) - \int_0^1 dt\, \rho'(s)(H^\beta - H^\alpha)(u(s,t)) + \rho'(s)(S^\beta - S^\alpha)(u(s,0)) \right\} \le$$

$$\le \varepsilon - \int_0^1 \min(H^\beta - H^\alpha)dt + \max(S^\beta - S^\alpha)$$

Letting ε tend to 0 implies the theorem, since it is true for $\varepsilon^{\alpha,\beta}_{(H,S)} + \varepsilon$ for every $\varepsilon > 0$. Before concluding we would like to make the last estimation clearer: given the formula for the differential of $\mathcal{A}_{(H,S)}$ and that u satisfies to the conditions (3.9), it is easy to see that

$$d_{u(s,\cdot)}\mathcal{A}_{(H_s,S_s)}.\partial_s u(s,\cdot) = -\int_0^1 \omega(\partial_s u(s,t), J^{\alpha,\beta}\partial_s u(s,t))dt < 0$$

whose integral is finite, since the energy of u is (see (3.10)). □

Now that we know how the homotopy does not preserve the filtration, we want to check that it can give us the Morse homology groups of the generating function and the Floer homology groups of the action functional.

Let us start with the latter: as almost complex structure we choose $J \oplus i = (J_t \oplus i)_{t\in I}$ on $T^*E = T^*M \oplus \mathbb{C}^k$, we keep the target Hamiltonian, $H \oplus 0 : T^*M \oplus \mathbb{C}^k \to \mathbb{R}$, but we suppose the generating function to be a constant quadratic form $S = Q_0$ on $E = M \times \mathbb{R}^k$ (this is possible since E is trivial, S smooth and the space of non degenerate quadratic forms with a fixed signature is contractible by Gram-Schmidt). Given the product structure in both the symplectic form and the almost complex structure, conditions (3.9)[3]split into two components, $u : \mathbb{R} \times I \to T^*M$, $v : \mathbb{R} \times I \to \mathbb{C}^k$ satisfying to

$$\begin{cases} \overline{\partial}_{J,H} u = 0 \\ u(s,1) \in \nu^*N, \;\; u(s,0) \in 0_M \end{cases} \tag{3.13}$$

$$\begin{cases} \overline{\partial} v = 0 \\ v(s,1) \in 0_{\mathbb{R}^k} = \mathbb{R}^k \subset \mathbb{C}^k, \;\; v(s,0) \in \mathrm{Graph}(dQ_0) \end{cases} \tag{3.14}$$

The solutions of (3.14) are particularly simple: the first equation says that v is holomorphic, and in this context actually constant. Since $S = Q_0$ is a generating function, $\mathrm{Graph}(dQ_0) \pitchfork \mathbb{R}^k$, then $\mathrm{Graph}(dQ_0) \cap \mathbb{R}^k = 0$ as $\mathrm{Graph}(dQ_0)$ is a linear subspace of $\mathbb{C}^k$ of real dimension k. Summing up, v is a holomorphic strip with Lagrangian boundary conditions, with finite energy tending to 0 at infinity: the only solutions v satisfying these constraints are constant ([25]).

This proves that the only conditions on the solutions which actually matters is (3.13), which is the same system of equations that defines Floer trajectories: the moduli space $\mathcal{M}_{(J,H,Q)}(N : E)$ is the same as $\mathcal{M}(J,H,N : E)$ we defined in Section 3.3. Moreover, since the component on $\mathbb{C}^m$ of $X_H = 0$ is trivially 0, critical points of $\mathcal{A}_{(H,S)}$ coincide with those of $\mathcal{A}_H$, and the values are the same, under the correspondence $\gamma \leftrightarrow (\gamma, 0)$. This establishes

$$HF^\lambda_\bullet(J,H,Q,N : E) \cong HF^\lambda_\bullet(J,H,N : M) \tag{3.15}$$

[3] J, S, H here do not depend on s, so these conditions describe integral curves for $-\mathrm{grad}\mathcal{A}_{(H,mS)}$ as mentioned above.

We now want to represent the Morse complex of S_N we defined in Section 3.2. Here we pick $H = 0$, the generating function $S : E \to \mathbb{R}$, and a family of complex structures $(J_t)_{t \in I}$: this induces a family of complex structures on T^*E given by

$$J_{S,t} = \left(\phi^t_{\pi^* S}\right)_* (J_t \oplus i)$$

for $t \in I$ and $\pi : T^*E \to E$ the canonical projection. In this setting, the integral lines of $-\mathrm{grad}\mathcal{A}_{(0,S)}$ in $\mathcal{P}(S, N : E)$ satisfy to:

$$\begin{cases} \overline{\partial}_{J_S} U = 0 \\ U(s,0) \in \mathrm{Graph}(dS), \ U(s,1) \in \nu^* N \times 0_{\mathbb{R}^m} \end{cases} \tag{3.16}$$

and remark that they connect actual intersections between the graph of dS and the conormal (since the Hamiltonian is 0, the Hamiltonian paths are points): this implies that we already are looking at the Morse complex of S. We need to check that the differentials of the two theories coincide.

At first, take a small tubular neighbourhood $\mathcal{U}$ of N in M, say of thickness ε (with respect to a chosen Riemannian metric on M); let $\pi_N : \mathcal{U} \to N$ be the associated projection. We can deform S in S', another generating function, in a way that:

$$S' = \begin{cases} S & \text{outside } \mathcal{U} \text{ and a compact set in} E \\ S_N \circ \pi_N & \text{on } E|_{\mathcal{U}} \end{cases}$$

The aim of this is to make the gradient field tangent to N, so that there are no Morse trajectories crossing N. This deformation can clearly be made arbitrarily small in the $\mathcal{C}^0$ topology, since it is of order ε. A priori, we cannot make it small in the $\mathcal{C}^1$ norm. Up to replacing S or S' with an arbitrarily close generating function, we have an isomorphism

$$HF^\lambda_\bullet(J, 0, S, N : E) \simeq HF^{\lambda+\varepsilon_{0,S}}_\bullet(J, 0, S', N : E)$$

by Theorem 3.4.5, and $\varepsilon_{0,S}$ is of order $\varepsilon \to 0$.

Through some analysis (see [23]), one could find an isomorphism

$$HF_\bullet(J, 0, S', N : M) \simeq HM_\bullet(S', N : E) \tag{3.17}$$

and this isomorphism also preserves critical levels of $\mathcal{A}_{(0,S')}$ and S'. The idea, as explained in [11], [28], and [24], is the following: using the hamiltonian flow of $\pi^* S'$, one can the define a bijection between the relevant moduli spaces. In particular, if $u : \mathbb{R} \to E$ is a Morse gradient-line, then

$$\tilde{u}(s,t) = \phi^{1-t}_{\pi^* S'} u(s)$$

satisfies (3.16). The inverse of this correspondence is the obvious one, what is not obvious is the good definition: if u is a Floer curve for (3.16), one has to show that $\tilde{u}(s) = (\phi^{1-t}_{\pi^* S'})^{-1} u(s,t)$ does indeed not depend on the time t. To do so, one define an "energy"

$$f(s) = \int_0^1 \|y(s,t)\|^2 dt$$

where $\tilde{u} = (x, y)$ in the cotangent canonical coordinates. f is non-negative and tends to 0 at $\pm\infty$; analytical arguments in [24] show that it is also convex, hence constantly 0, as soon as the $\mathcal{C}^2$ norm of S' is small enough: we have a bijection of the moduli spaces. To make this norm small enough, it is in fact sufficient to rescale the Riemannian metric one uses in the construction (this does not influence the good definition, or not, of the complexes).

Now, by definition of S', $S_N = S'_N$: the two Morse complexes coincide, and the filtration is respected once again. Composing the isomorphisms, we have that for every $\lambda \in \mathbb{R}$

$$HF^{\lambda}_{\bullet}(J, 0, S, N : M) \simeq HM^{\lambda+\varepsilon_{0,S}}_{\bullet}(S, N : E)$$

Since $\varepsilon_{(0,S)}$ is arbitrarily small, if λ is not a critical value for S_N, which in this case is equivalent to $\lambda \notin \mathrm{Spec}(0, S, N : E)$, then the isomorphisms really preserve the filtration, as the action spectrum is compact and nowhere dense. We just quickly explain the identification between $\mathrm{Crit}(S_N)$ and $\mathrm{Spec}(0, S, N : E)$: since $H = 0$, the only critical points of $\mathcal{A}_{(0,S)}$ are constant curves. This implies that they are points which lie in the intersection $\mathrm{Graph}(dS) \cap \nu^*(E_N)$: this means exactly that they are critical points of S_N, and clearly we have the wanted bijection. Since the curves are constant and the Hamiltonian 0, computing $\mathcal{A}_{(0,S)}$ at a critical point is the same as computing S on the same point, hence the correspondence.

3.4.3 Proof of Theorem 3.4.1

What we actually need to prove here is that for a $\lambda \notin Spec(L : N)$ we can make the ε of Theorem 3.4.5 as small as we want, so that it preserves the filtration (the action spectrum is compact).

If $L = L_S$, consider a family of generating functions $(S_t)_{t \in I}$ such that:

- $S_0 = S$;
- $S_1 = Q$ non degenerate quadratic form on the fibres;
- $L_{S_t} = (\phi^t_H)^{-1} L_S$.

Consider now a path of Hamiltonians $t \mapsto H_t$ such that $\phi^1_{H_t} = \phi^t_H$. Then for all t $\phi^1_{H_t}(L_{S_t}) = L_S$. Since the action spectrum depends on the Lagrangian submanifold only (up to a normalisation; see [27]), $\mathrm{Spec}(H(t) \oplus 0, S_t, N : E)$ is constantly equal to $\mathrm{Spec}(H(0) \oplus 0, S_0, N : E) = \mathrm{Spec}(0, S, N : E)$, which is finite as it coincides with the set of critical values of S_N, $\mathrm{Spec}(L : N)$.

Let $\varepsilon > 0$ be the minimum distance between two distinct points in the action spectrum. The path $t \mapsto (H_t, S_t)$ is smooth by definition, therefore there is a small $\delta > 0$ such that

$$\|H_s - H_t\|_{\mathcal{C}^0} + \|S_s - S_t\|_{\mathcal{C}^0} < \frac{\varepsilon}{3}$$

as soon as $|s - t| < \delta$. In particular, for such t and s, the isomorphisms in Theorem 3.4.5 take the form (here we write H instead of $H \oplus 0$, and J instead of $J \oplus i$)

$$h_{st} : HF_\bullet^\lambda(J, H_s, S_s, N : E) \to H_\bullet^{\lambda + \frac{\varepsilon}{3}}(J, H_t, S_t, N : E)$$
$$h_{ts} : HF_\bullet^\lambda(J, H_t, S_t, N : E) \to H_\bullet^{\lambda + \frac{\varepsilon}{3}}(J, H_s, S_s, N : E)$$

and their composition becomes

$$h_{st} \circ h_{ts} : HF_\bullet^\lambda(J, H_s, S_s, N : E) \to H_\bullet^{\lambda + \frac{2\varepsilon}{3}}(J, H_s, S_s, N : E)$$

By definition of ε, however,

$$H_\bullet^{\lambda + \frac{\varepsilon}{3}}(J, H_s, S_s, N : E) = H_\bullet^{\lambda + \frac{2\varepsilon}{3}}(J, H_s, S_s, N : E) = H_\bullet^\lambda(J, H_s, S_s, N : E) \tag{3.18}$$

given that the two underlying chain complexes and chain maps are exactly the same. Denote then, using the identity (3.18), $h_{st}^\lambda, h_{ts}^\lambda$ the induced morphisms which preserve the filtration. Note that, whereas they are isomorphisms for the whole homology groups, they need not be, a priori, be isomorphisms for the filtration; one could check however that $h_{st}^\lambda \circ h_{ts}^\lambda = h_{tt}^\lambda = Id$, for any $\lambda \notin Spec(L : N)$, so isomorphisms they are. The idea is to concatenate them properly.

Consider a partition of the unit interval

$$0 = t_0 < t_1 < \cdots < t_k = 1$$

which is finer than δ. Then for every $\lambda \notin Spec(L : N)$, for every $j \in \{0, \ldots, k-1\}$, $h_{t_j, t_{j+1}}^\lambda$ is an isomorphism preserving the filtration of inverse h_{t_{j+1}, t_j}^λ. Considering compositions of the kind

$$h_{t_{k-1}, t_k}^\lambda \circ \cdots \circ h_{t_0, t_1}^\lambda : HF_\bullet^\lambda(J, 0, S, N : E) \to HF_\bullet^\lambda(J, H, Q, N : E)$$

where λ may be ∞, using k times Theorem 3.4.5 we get to the commutative diagram:

$$\begin{array}{ccc} HF_\bullet^\lambda(J, H, Q, N : E) & \longrightarrow & HF_\bullet^\lambda(J, 0, S, N : E) \\ \downarrow j_*^\lambda & & \downarrow j_*^\lambda \\ HF_\bullet^\lambda(J, H, Q, N : E) & \longrightarrow & HF_\bullet(J, 0, S^\alpha, N : E) \end{array}$$

Using the filtration-preserving canonical isomorphisms (3.15) and (3.17), we establish the Theorem.

Appendix A

Chern-Weil Theory

This section is based on the first chapters of [34] for the definition of characteristic classes, [5] for the necessary obstruction theory, and some lecture notes to link the two approaches. Manifolds and bundles we consider here shall be complex, but vector bundles may not be holomorphic. We are going to define the notion of characteristic class, and most notably we are going to explain what the Chern class is, and to which formal properties it satisfies. The decision of splitting the Appendix into two parts comes from the fact that the first, except for the last part, is quite different in nature from the second part. Also, Chern-Weil theory, albeit only touched in this text and for utilitarian reasons, still deserves a space of its own.

A.1 The Chern-Weil Theorem and the Chern class

We omit the standard definitions of connections on a vector bundle and of its curvature.

Let M be a smooth manifold, E a vector bundle on M, $\mathrm{End}(E)$ the endomorphism bundle of E. For a section $A \in \Gamma(\mathrm{End}(E))$ we can define its trace pointwise. This induces another trace operator on $\Omega^k(M; \mathrm{End}(E))$ the obvious way: $\omega \otimes A \mapsto \mathrm{tr}[A]\omega$. $\Omega^\bullet(M)$ and $\Omega^\bullet(M; \mathrm{End}(E))$ are both superalgebras, and in particular the latter in endowed with the supercommutator

$$[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes (A \circ B) - (-1)^{|\omega||\eta|}(\eta \wedge \omega) \otimes (B \circ A)$$

Lemma A.1.1. $\mathrm{tr}([\cdot, \cdot]) = 0$.

Proof. Clear as $\mathrm{tr}(A \circ B) = \mathrm{tr}(B \circ A)$ and $\omega \wedge \eta = (-1)^{|\omega||\eta|}\eta \wedge \omega$. □

Lemma A.1.2. For $A \in \Omega^\bullet(M; \mathrm{End}(E))$ one has the equality

$$d(\mathrm{tr}[A]) = \mathrm{tr}[\nabla^E, A]$$

for any choice of the connection ∇^E. The term on the right-hand side is to interpreted as follows:

$$[\nabla^E, A]s = \nabla^E(As) - (-1)^{|A|} A \wedge (\nabla^E s)$$

for any $s \in \Gamma(E)$.

Proof. For the statement to have a meaning, one needs to prove that the right-hand side really does not depend on the choice of specific connection we make. Let ∇, ∇' be two connections on E. Then their difference is clearly $\mathscr{C}^\infty(M)$-linear, and therefore an element of $\Omega^\bullet(M; \mathrm{End}(E))$: by previous lemma then $\mathrm{tr}[\nabla - \nabla', A] = 0$, and we have the desired independence. For the left-hand side, every operation we make is local, so it suffices to check the identity on trivialising sets for the bundle $E \to M$. On a trivialising open set however we can suppose the connection to be trivial since the right-hand side does not depend on such assumption. More explicitly: let (s_i) be a smooth local basis for E: then if $\omega \in \Omega^\bullet(M; E)$ we can locally write it as $\omega = \eta^i \otimes s_i$ for some $\eta^i \in \Omega^\bullet(M)$. We therefore define ∇^E as $\nabla^E s = (d\eta^i) \otimes s_i$(note that it may not be possible to extend this connection globally). We can also similarly locally write A as $A = \alpha \otimes B$. But then since

$$[\nabla^E, A]s_i = \nabla^E(\alpha \otimes Bs_i) - (\alpha \wedge \nabla^E s_i) \otimes B = \nabla^E(\alpha \otimes Bs_i)$$

as ∇^E is 0 on the basis by definition, we conclude that locally the equality needs to be true for the trivial connection, and therefore for every connection (which might be globally defined). □

For a connection ∇^E, we define its curvature $R^E = \left(\nabla^E\right)^2$. Let us now then consider a function $f : U \to \mathbb{C}$ which is analytic near 0 (U open domain in $\mathbb{C}$), and let $f(x) = \sum a_n x^n$ its power series. We consider then the series

$$f(R^E) = \sum a_n \left(R^E\right)^n \in \Omega^{2\bullet}(M; \mathrm{End}(E))$$

where the exponentiation stands for repeated composition. If the dimension of the manifold is finite, the sum is finite too.

Theorem A.1.3 (Chern-Weil)**.** $\mathrm{tr}[f(R^E)]$ defines a cohomology class on M which does not depend on the chosen connection ∇^E.

Proof. At first we prove that the defined form is in fact closed. But using Lemma A.1.1 and $[\nabla^E, (R^E)^k] = 0$ (Bianchi's identity) we have, since the sum is finite

$$d\,\mathrm{tr}[f(R^E)] = \sum a_n \,\mathrm{tr}[\nabla^E, (R^E)^n] = 0$$

We now need to prove that if ∇_0^E, ∇_1^E are two connections on E, R_0^E, R_1^E the respective curvatures, then $\mathrm{tr}[f(R_0^E)] - \mathrm{tr}[f(R_1^E)] = d\omega$, for some $\omega \in \Omega^\bullet(M)$. We start by defining the connection $\nabla_t^E = t\nabla_1^E + (1-t)\nabla_0^E$. Then by the Fundamental Theorem of Calculus

$$\mathrm{tr}[f(R_1^E)] - \mathrm{tr}[f(R_0^E)] = \int_0^1 \frac{d}{dt} \,\mathrm{tr}[f(R_t^E)]\, dt$$

We need to show that the right-hand side is exact. Now, applying the linearity of the trace, the chain rule and Bianchi's identity, if $f'(x)$ denotes the derived series of f, we have

$$\frac{d}{dt}\operatorname{tr}[f(R_t^E)] = \operatorname{tr}\left[\frac{d}{dt}f(R_t^E)\right] = \operatorname{tr}\left[f'(R_t^E)((\nabla_1^E - \nabla_0^E)\nabla_t^E + \nabla_t^E(\nabla_1^E - \nabla_0^E))\right]$$
$$= \operatorname{tr}[f'(R_t^E)[\nabla_1^E - \nabla_0^E, \nabla_t^E]] = \operatorname{tr}[f'(R_t^E)(\nabla_1^E - \nabla_0^E), \nabla_t^E] = d\operatorname{tr}[f'(R_t^E)(\nabla_1^E - \nabla_0^E)]$$

In these lines we also Lemma A.1.1 and that $\nabla_1^E - \nabla_0^E \in \Omega^1(M; \operatorname{End}(E))$, hence the equality:

$$(\nabla_1^E - \nabla_0^E)\nabla_t^E + \nabla_t^E(\nabla_1^E - \nabla_0^E) = [\nabla_1^E - \nabla_0^E, \nabla_t^E]$$

from the definition in Lemma A.1.1. We conclude interchanging differential and integral. □

We now define the total Chern class using Theorem A.1.3. Let E be a complex vector bundle on M and ∇^E a connection on E of curvature R^E.

Definition A.1.1 (Total Chern form/class)**.** We define the total Chern form to be:

$$c(E, \nabla^E) = \det\left(Id + \frac{i}{2\pi}R^E\right)$$

where Id is the identity endomorphism of E. The total Chern class is the associated cohomology class.

One has the equality

$$\det\left(Id + \frac{i}{2\pi}R^E\right) = \exp\left(\operatorname{tr}\left[\log\left(Id + \frac{i}{2\pi}R^E\right)\right]\right)$$

log is analytic around 1 and exp on the whole of $\mathbb{C}$, so that by Theorem A.1.3 the definition makes sense: the total Chern form is in fact closed, and the total Chern class is (well defined and) not dependent on the choice of the connection ∇^E. We shall therefore write $c(E)$ for $c(E, \nabla^E)$. We can define the *i-th Chern class* to be the i-th term of the sum, $[c_i(E)] \in H^{2i}(M; \mathbb{C})$. Remark that $[c_1(E)] = \frac{i}{2\pi}\operatorname{tr}[R^E]$ as one would expect: in the case of line bundles this definition restricts to the usual one.

A.2 Axioms for the Chern class

The aim of this section is to prove the following characterisation of the Chern class:

Theorem A.2.1. There is a unique application sending a complex vector bundle E on M to a cohomology class $c(E) \in H^{2\bullet}(M; \mathbb{C})$ satisfying the following four requirements:

i) $c(E) = 1 + c_1(E) + \dots$, with $c_i \in H^{2i}(M;\mathbb{C})$;

ii) $c(E \oplus F) = c(E) \wedge c(F)$;

iii) If $f : V \to M$ is continuous, $f^*c(E) = c(f^*E)$;

iv) For a line bundle $L \to M$, $c(L) = 1 + \frac{i}{2\pi}[R^L]$.

Once we show this theorem, we can check that the Chern class we defined satisfies to the four axioms. Axioms i) and iv) are clear; axiom ii) is verified as $R^{E\oplus F} = R^E \oplus R^F$; axioms iii) comes from the fact that $R^{f^*E} = f^*R^E$.

For the proof of this theorem, we admit the following lemma:

Lemma A.2.2 (Splitting Principle)**.** Let $E \to M$ be a complex vector bundle. Then there exists a manifold P together with a map $\pi : P \to M$ such that:

i) $\pi^* : H^\bullet(M;\mathbb{C}) \to H^\bullet(P;\mathbb{C})$ is injective;

ii) $\pi^*E = \bigoplus_i L_i$, for some line bundles L_i.

To prove Theorem A.2.1, we clearly only need to prove the uniqueness part: the existence is trivial since we showed that at least the correspondence complex bundle→its total Chern class is an exemple of such application.

Let us then approach the uniqueness: for a complex vector bundle $p : E \to M$, consider the map of manifolds $\pi : P \to M$ whose existence and properties are the content of the Splitting Principle. Then:

$$\pi^*c(E) = c(\pi^*E) = c\left(\bigoplus L_i\right) = \bigwedge c(L_i) = \bigwedge\left(1 + \frac{i}{2\pi}[R^{L_i}]\right)$$

that is, by injectivity of π^*, c is entirely determined by its definition on the line bundles.

Remark. There is a second point of view on the first Chern class, which is the following: we start by defining the Picard group of M, whose elements are (isomorphism classes of) line bundles on M, the binary operation being the tensor product over $\mathbb{C}$, so that the inverse of a line bundle is its dual (if $\dim_\mathbb{C} V = \dim_\mathbb{C} W = 1$, then $V^* \otimes W = \mathrm{Hom}(V, W) \simeq \mathbb{C}$). One can show that $Pic(M)$ is isomorphic to $H^1(M;\mathcal{O}^*_M)$, the Čech first cohomology group of the sheaf of holomorphic functions on M which are never 0. We then define the sheaf of holomorphic functions on M, $\mathcal{O}_M$ and the sheaf of locally constant functions with values in $\mathbb{Z}$ on M. Using the short exact sequence of sheaves:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_M \to \mathcal{O}^*_M \to 0$$

called *exponential sequence*, we get the Bockstein homomorphism $\delta : H^1(M;\mathcal{O}^*_M) \to H^2(M;\mathbb{Z})$ (the cohomology with values in the locally constant sheaf $\mathbb{Z}$ is the singular cohomology of the manifold). One can prove, by means of spectral sequences, that the composition $Pic(M) \to H^1(M;\mathcal{O}^*_M) \xrightarrow{\delta} H^2(M;\mathbb{Z})$ is precisely the assignment of the first Chern class as we defined it. In particular, one finds that the first Chern class is in fact a class of integer cohomology: integrated on a generator of $H_2(M;\mathbb{Z})$ it gives an integer.

Remark. Note that the uniqueness in Theorem A.2.1 would make the second part of the statement of Theorem A.1.3 obsolete. However, we cannot omit the latter, otherwise the statement of the former loses its meaning.

A.3 Chern class and obstruction

We now need to prove the following Theorem:

Theorem A.3.1. Let $p : E \to B$ be a complex vector bundle of (complex) rank n. Then the top Chern class $c_n(E) \in H^{2n}(B;\mathbb{R})$ vanishes.

Why are we interested in proving it? For complex vector bundles $E, F \to B$ we have that $c(E \otimes F) = c(E) + c(F)$: the reason is that $R^{E\otimes F} = R^E \otimes Id_F + Id_E \otimes R^F$. In particular if there is a global non vanishing section of $E \otimes E \to M$ (for E line bundle) then the first Chern class needs to be 2-torsion. Now, for a manifold M^n we can apply this result to $E = \bigwedge_{\mathbb{C}}^n T^*M$: the tangent space of a lagrangian submanifold of T^*M being of dimension n, if there is a global section ϑ of $E^{\otimes 2}$, taking the angle $arg(\vartheta)$ (it is here where we need ϑ not to vanish) gives a function $Gr(Lag(T^*M)) \to \mathbb{S}^1$, which is a first step in the definition of a Maslov index. See [3] for a brief discussion about this point. The result of this informal argument is that $2c_1(T^*M) = 0$ is a necessary condition to define a $\mathbb{Z}$-grading on the Lagrangian Floer complex (the first Chern class of the cotangent bundle is the same as the one of its determinant bundle: see [13]).

To prove the Theorem, we need to introduce the projective of a (complex) vector bundle: essentially, it is the vector bundle which has as fibre the projective space of the initial bundle.

Definition A.3.1 (Projective bundle)**.** Let $p : E \to M$ be a complex vector bundle, with the trivialising cover $(\mathcal{U}_\alpha)$ of M and transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL_n(\mathbb{C})$. The projective bundle of E, $P(E)$, is then defined as the bundle with fibre $P(E_x)$ at $x \in M$, where $E_x = p^{-1}(x)$, trivialisations given by the same cover $\mathcal{U}_\alpha$ and transition maps induced by the $\varphi_{\alpha\beta}$.

Remark. The projective bundle has a key role in the proof of the Splitting Principle, see [5].

We can now prove the Theorem above:

Proof. Let s be a non vanishing section of $E \to M$: it induces canonically a section of the projective bundle $P(E) \to M$. Let us consider the bundle

$$S_E = \{\, (l, v) \in P(E) \times E \mid v \in l \,\}$$

This is called the **universal subbundle** of $\hat{E} = P(E) \times E$ on $P(E)$. Then the line bundle $\tilde{s}^*S_E$ whose fibre at $x \in M$ is the vector space spanned by $\tilde{s}(x)$ in E_x is tautologically trivial (the global trivialisation is given by definition by $\tilde{s}$). By naturality of the Chern class, $\tilde{s}^*c_1(S_E) = 0$ (to see that the Chern class of a trivial bundle is 0, one can just take the trivial connection on it;

on dimension 1 we can use the remark that $c_1 : Pic(M) \to H^2(M;\mathbb{Z}))$. Let $\pi : P(E) \to M$ be the projection of the projective bundle. One can prove, using Leray-Hirsch Theorem, that $H^\bullet(P(E))$ is a free $H^\bullet(M;\mathbb{Z})$-module, where a basis is given by $\{1, x, \ldots, x^{n-1}\}$ for $x = \pi^*(c_1(E))$ (the multiplications are wedge products). Yet another possible definition for the $i-$th Chern class is the coefficient multiplying x^{n-i} in the summation

$$x^n = a_n + a_{n-1}x^1 + \cdots + a_1 x^{n-1}$$

Therefore, applying $\tilde{s}^*$ to the above expression, we see that $\tilde{s}^*\pi^* c_n(E) = c_n(E) = 0$, that which we wanted to show. $\square$

Appendix B

Chern and Maslov Classes

Here we are going to prove that there exists a Maslov class on a symplectic manifold if and only if the Chern class has torsion 2, following [33]. Before proceeding with the proof, we are going to introduce the objects and tools that are going to play a role in it, namely classifying spaces and spectral sequences.

B.1 Universal bundles and classifying spaces

For the proofs of this section we refer to the book [26]. We shall nevertheless try to convey the main involved ideas. We shall consider only real vector bundles, but the complex case is completely analogue. A universal bundle is basically a bundle from which we can obtain, via pull-back, all the others. To define it, and prove that it has indeed the desired properties, we need to start with the Grassmannian manifold of subspaces of $\mathbb{R}^{n+k}$: it will give us a way to find good bundles with similar, but weaker, properties to the one we want to get.

Definition B.1.1 (Grassmann manifold)**.** The Grassmann manifold $Gr_n(\mathbb{R}^{n+k})$ is a manifold whose points are the n-dimensional vector subspaces of $\mathbb{R}^{n+k}$.

$Gr_n(\mathbb{R}^{n+k})$ is indeed a manifold, and smooth. Its topology is defined as the quotient topology for the map $V_n(\mathbb{R}^{n+k}) \to Gr_n(\mathbb{R}^{n+k})$, where $V_n(\mathbb{R}^{n+k})$ is the *Stiefel manifold* of the $n-$frames (collections of n independent vectors) in $\mathbb{R}^{n+k}$. The Stiefel manifold is an open subset of $\mathbb{R}^{n(n+k)}$, hence inherits a (smooth) differential structure. Identifying the frames generating the same subspaces, we have the Grassmannian manifold. One can prove that it is indeed a topological manifold (second countable, Hausdorff, locally euclidean), and requiring the projection $V_n(R^{n+k}) \to Gr_n(\mathbb{R}^{n+k})$ to be (smoothly) differentiable, we achieve our goal: $Gr_n(\mathbb{R}^{n+k})$ is a (smoothly) differentiable manifold of dimension nk.

We can define a tautological vector bundle on $Gr_n(\mathbb{R}^{n+k})$ attaching to every point of $Gr_n(\mathbb{R}^{n+k})$, hence to a vector subspace of $\mathbb{R}^{n+k}$, itself. Again, one could prove that the result is locally trivial, and we denote the obtained vector bundle with $\gamma^n(\mathbb{R}^{n+k}) \to Gr_n(\mathbb{R}^{n+k})$.

Now, we can notice that given an embedded manifold $M^n \subseteq \mathbb{R}^{n+k}$, we have a morphism of vector bundles given by $TM \to \gamma^n(\mathbb{R}^{n+k})$, $(x, v) \mapsto (T_x M, v)$: via the embedding $M \hookrightarrow \mathbb{R}^{n+k}$ we can identify the tangent space at every point of the manifold to a subspace of $\mathbb{R}^{n+k}$. we have the clear commutative diagram

$$\begin{array}{ccc} TM & \longrightarrow & \gamma^n(\mathbb{R}^{n+k}) \\ \big\downarrow & & \big\downarrow \\ M & \longrightarrow & Gr_n(\mathbb{R}^{n+k}) \end{array}$$

where the vertical arrows are the bundle projections, the upper horizontal arrow is the one we already defined, the lower one is simply the correspondence $x \mapsto T_x M$.

We can extend the idea to more general vector bundles or rank n, but k might need to increase.

Lemma B.1.1. *Let $\xi \to B$ be a vector bundle of rank n, with B compact. Then there is a bundle morphism $\xi \to \gamma^n(\mathbb{R}^{n+k})$ for a large enough k.*

We shall omit a detailed proof, however the idea is pretty simple: one uses a finite trivialising cover of B and partitions of unity to apply the line of reasoning we explained above. In particular the k we add depends on the cardinality of such an open trivialising cover[1]. Hence the idea of taking the inductive limit of the Grassmannians to create a bundle, which is universal in the sense that we can always find a map from any vector bundle of rank n to this bundle, and the basis can be compact or even just paracompact (from any open cover one can take a countable subcover).

Definition B.1.2 (Infinite Grassmannian manifold)**.** We define $Gr_n^{\mathbb{R}} := Gr_n(\mathbb{R}^\infty)$ as the set of linear subspaces of $\mathbb{R}^\infty$ (the set of sequences in $\mathbb{R}$ with only a finite number of non-zero terms) with the topology making all the canonical injections $Gr_n(\mathbb{R}^{n+k}) \hookrightarrow Gr_n^{\mathbb{R}}$ continuous (so $G_n^{\mathbb{R}} = \varinjlim G_n(\mathbb{R}^{n+k})$).

It is indeed a manifold (see for instance [12]), over which we can define, again, a tautological bundle $\gamma^{n,\mathbb{R}}$ attaching to a point itself as a vector subspace of $\mathbb{R}^\infty$. This bundle is indeed a bundle (a bit longer to prove than before, since it requires a small technical lemma about the product topology of the inductive limit), and its universality is the content of the next theorem:

Theorem B.1.2. *For any $n \in \mathbb{N}$, for any vector bundle $\xi \to B$ of rank n with a paracompact base, there is a bundle morphism $\xi \to \gamma^n$. Moreover, two such morphisms from the same bundle $\xi \to B$ are homotopic.*

Proof. It can be found in pages 65 to 68 of the book by Milnor and Stasheff. □

[1]There is a great similarity in fact between the proof of this theorem and the proof of the (weak) Whitney's Embedding Theorem: there the open trivialising cover for the tangent bundle is simply an atlas, and by compactness we can suppose it to be finite.

From here on we shall consider both the complex and the real case, using the same notation as above and replacing $\mathbb{R}$ with $\mathbb{C}$ when necessary. All the previous results still hold in the complex case. We define the notion of classifying space for a topological group.

Definition B.1.3 (Classifying space)**.** Let G be a topological group. A classifying space for G is a G-principal bundle $W \to G\backslash W$ where W is weakly contractible, i.e. all the homotopy groups of W are 0.

From now on, the Stiefel manifolds $V_n^{\mathbb{C}}$ and $V_n^{\mathbb{R}}$ will be the sets of unitary, or orthonormal, n-frames in $\mathbb{K}^\infty$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The constructions we did still hold, even though some details will be different. We omit the proof that they are both weakly contractible. Also, we have a clear structure of principal $U(n)$-bundle for $V_n(\mathbb{C}^{n+k}) \to Gr_n(\mathbb{C}^{n+k})$, since we quotient according to the left action of $U(n)$. Similarly we have a principal $O(n)$-bundle in $V_n(\mathbb{R}^{n+k}) \to Gr_n(\mathbb{R}^{n+k})$. Moreover, given the canonical inclusions $V_n(\mathbb{C}^{n+k}) \hookrightarrow V_n(\mathbb{C}^{n+k+1})$ and $V_n(\mathbb{R}^{n+k}) \hookrightarrow V_n(\mathbb{R}^{n+k+1})$, the actions of $U(n)$ and $O(n)$ are compatible with such maps. This implies (we omit the technical details) that the actions are compatible with the passage to the direct limits, hence we finally have the two principal G-bundles $V_n^{\mathbb{C}} \to Gr_n^{\mathbb{C}}$ and $V_n^{\mathbb{R}} \to Gr_n^{\mathbb{C}}$, which then give rise to two classifying spaces for $U(n)$ and $O(n)$ respectively. We can now prove the property we are really interested in applying:

Proposition B.1.3. Let $\pi : E \to P$ be a smooth $\mathbb{K}$-vector bundle ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, the proof is the same) of rank n on which G acts fibrewise ($G = U(n)$ in the complex case and $G = O(n)$ in the real one) where P is a smooth manifold. Then there is a smooth function $f : P \to Gr_n^{\mathbb{K}}$ such that $E \simeq f^* V_n^{\mathbb{K}}$.

Proof. Let $q : V_n^{\mathbb{K}} \to Gr_n^{\mathbb{K}}$ denote the projection. From here on, we drop the $\mathbb{K}$ from the notation. One can define a map $\psi : E \to V_n$ similar to the ones one defines in the proofs of Lemma B.1.1 and of its paracompact analogue: choosing local trivialisations we can give at every point $x \in P$ a frame for E_x, globalising the construction using partitions of unity. We then find the map $\phi = q \circ \psi$, which is in fact a local identification of the fibres of E with subspaces of $\mathbb{K}^\infty$. Then, since the image of ϕ only depends on the fibres and not on the particular vector, there is one and only map $f : P \to Gr_n$ such that $f \circ \pi = \phi$. We sum the information up in the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & V_n \\ \downarrow{\scriptstyle\pi} & \searrow^{\phi} & \downarrow{\scriptstyle q} \\ P & \xrightarrow{f} & Gr_n \end{array}$$

We remark that since G acts fibrewise, for any $g \in G$, $q(g\cdot) = q$, $\pi(g\cdot) = \pi$, and the action of G on E is a lift of the trivial action of G on P: then $f(g\cdot) = f$. We now prove that $E \simeq f^* V_n$: given the definition

$$f^* V_n = \{ (x, \sigma) \in P \times V_n \mid f(x) = q(\sigma) \}$$

we see that in fact f^*V_n is nothing but the local data of a point in $x \in P$ and an n-frame of $\mathbb{K}^\infty$ representing the fibre E_x in the chosen trivialisation, and the isomorphism is clear. Also, given the diagonal action of G on $P \times V_n$, and the above identities, we see that the action of G on E is indeed compatible with the one on the pull-back. $\square$

We mention here the crucial fact that a Grassmannian can be endowed with a CW-complex structure; once again we refer to [26] for the details.

B.2 Spectral sequences

Spectral sequences are a tool to calculate (co)homology groups of a chain complex. It generalises the well-known phenomenon of constructing a long exact sequence in the (co)homology from a short exact sequence of chain complexes, in a way we are going to see later. This section will be based on the notes of the course by Julien Marché [19]. Other standard references are [21] and [20]. In particular, we shall define the general concept of homological spectral sequence and see why the relative homology long exact sequence is indeed a special case of spectral sequence, and later we shall define the Leray-Serre spectral sequence. The cohomological version will follow immediately, and we will be able to recover a cup-product structure on the sequence.

Let $(C_n)_{n\in\mathbb{Z}}$ be a chain complex, and assume that every C_n is endowed with a filtration $\cdots \subseteq F_pC_n \subseteq F_{p+1}C_n \subseteq \dots$ respected by the differential $\partial : C_n \to C_{n-1}$, i.e. $\partial F_pC_n \subseteq F_pC_{n-1}$. This filtration therefore induces a filtration in homology:

$$F_pH_n(C_*) := \{ [x] \in H_n(C_*) \mid x \in F_pC_n \}$$

Let us also define a graduated complex: $G_pC_n := F_pC_n/F_{p-1}C_n$. It is still a complex since ∂ passes to the quotient: $G_pC_n \ni [x] \mapsto \partial[x] := [\partial x] \in G_pC_{n-1}$. We define thus the *zeroth page* of the spectral sequence of the complex C_*:

$$E^0_{p,q} := G_pC_{p+q}$$

with the boundary morphism $\partial_0 = \partial$. Let us define then the *first page* of the spectral sequence by

$$E^1_{p,q} = H_{p+q}(G_pC_*)$$

where the boundary morphism is the one induced by ∂_0: if $\alpha = [x] \in E^1_{p,q}$ is a homology class, we have that $x \in F_pC_{p+q}$ and $\partial_0 x \in F_{p-1}C_{p+q}$; we then set $\partial_1 : E^1_{p,q} \to E^1_{p-1,q}$ as $\partial_1[\alpha] = [\partial_0 x]$. We can proceed this way, taking the homology and computing the boundary morphism; there is however a closed form for the result: if we define the two vector spaces

$$A^r_{p,q} = \{ x \in F_pC_{p+q} \mid \partial_x \in F_{p-r}C_{p+q-1} \}$$
$$D^r_{p,q} = F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})$$

we can check (and it is part of the content of the following lemma) that the *r-th page* is $E^r_{p,q} = A^r_{p,q}/A^r_{p,q} \cap D^r_{p,q}$.

Lemma B.2.1. A homological spectral sequence has the three following properties:

i) ∂ induces a boundary morphism on the *r-th sheet* $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

ii) Indeed, the complexes on the *(r+1)-th* sheet are the homologies of the ones on the *r-th*:

$$E^{r+1}_{p,q} = \ker(\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{Im}(\partial_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$$

iii) If the filtration is bounded, i.e. there are $p_0, p_1 \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}$ $\forall p \leq p_0$ $F_pC_n = 0$, $\forall p \geq p_1$ $F_pC_n = C_n$, then there is an $r_0 \in \mathbb{N}$ such that $\forall r \geq r_0$, $E^r_{p,q} = G_pH_{p+q}(C_*) = F_pH_{p+q}(C_*)/F_{p-1}H_{p+q}(C_*)$.

Proof. *i*) The induced morphism is induced as we did above: if $\alpha = [x] \in E^r_{p,q}$, then we define $\partial_r\alpha = [\partial x]$. The good definition is an easy check, while the fact that $\partial_r^2 = 0$ is trivial.

ii) It is a long proof of linear algebra.

iii) Using the standard notations $Z_* = \ker \partial$ and $B_* = \operatorname{Im}\partial$, given the bounds on the filtration for large r we find $A^r_{p,q} = F_pZ_{p+q}$ and $D^r_{p,q} = F_{p-1}C_{p+q} + B_{p+q+1}$. Applying one of the Isomorphism Theorems for vector spaces and the identity $F_{p-1}Z_{p+q} = F_pZ_{p+q} \cap (F_{p-1}C_{p+q} + B_{p+q+1})$ we obtain the desired identity. □

We shall now examine the relative homology case. Let (X, A) be a pair of topological spaces. We have the two complexes $F_0C_* = C_*(A)$ and $F_1C_* = C_*(X) = C_*$. The graduation is then given by $G_0C_* = C_*(A) =: A_*$ and $G_1C_* = C_*/A_* =: B_*$. The zeroth page is then

$$\begin{array}{cc}
\vdots & \vdots \\
\downarrow{\scriptstyle \partial_0} & \downarrow{\scriptstyle \partial_0} \\
A_{q+1} & B_{q+1} \\
\downarrow{\scriptstyle \partial_0} & \downarrow{\scriptstyle \partial_0} \\
A_q & B_q \\
\downarrow{\scriptstyle \partial_0} & \downarrow{\scriptstyle \partial_0} \\
A_{q-1} & B_{q-1} \\
\downarrow{\scriptstyle \partial_0} & \downarrow{\scriptstyle \partial_0} \\
\vdots & \vdots
\end{array}$$

We pass to the second page taking the homologies of this complex, and we remark that ∂_1 is precisely the Bockstein homomorphism:

$$0 \longleftarrow H_q(A_*) \xleftarrow[\partial_1]{} H_{q+1}(B_*) \longleftarrow 0$$

$$0 \longleftarrow H_{q-1}(A_*) \xleftarrow[\partial_1]{} H_q(B_*) \longleftarrow 0$$

$$0 \longleftarrow H_{q-2}(A_*) \xleftarrow[\partial_1]{} H_{q-1}(B_*) \longleftarrow 0$$

The zeroes appearing to the left and to the right are the groups $E^1_{-1,\bullet}$ and $E^1_{2,\bullet}$ respectively. The chains would continue indefinitely to the left and to the right by zeroes, and up and down with shifted indices.

In page 2 we see that the only non-zero homology groups are $E^2_{0,q} = \operatorname{coker}(\partial_1 : H_{q+1}(B_*) \to H_q(A_*))$ and $E^2_{1,q} = \ker(\partial_1 : H_{q+1}(B_*) \to H_q(A_*))$, and by property *iii)* of Lemma B.2.1 we also have

$$\begin{aligned} E^2_{0,q} &= F_0 H_q(C_*) \\ E^2_{1,q} &= H_{q+1}(C_*)/F_0 H_{q+1}(C_*) \end{aligned}$$

This is equivalent to the exactness of the classic long exact sequence.

The Leray-Serre spectral sequence

Let $F \to (X, x_0) \xrightarrow{p} (B, b_0)$ be a Serre fibration with basepoints $b_0 \in B$, $x_0 \in X_{b_0}$ and $F = p^{-1}(b_0)$. Using the fibration property, we can make $\pi = \pi_1(B, b_0)$ act on $H_q(F; \mathbb{Z})$. Let us consider a singular n-simplex $\sigma \in C_n(F)$ and a curve γ representing a class $[\gamma] \in \pi$, and denote with Δ^n the elementary n-simplex in $\mathbb{R}^{n+1}$. Fix once and for all a homeomorphism $I^n \simeq \Delta^n$, so that we can apply the homotopy lifting property: we identify Δ^n with the lower face of a prism $\Delta^n \times \{0\}$, and we define the map $\Delta^n \times I \to B$, $(x, t) \mapsto \gamma(t)$. We then complete the square:

$$\begin{array}{ccc} \Delta^n \times \{0\} & \longrightarrow & X \\ \big\downarrow & \overset{\tau}{\nearrow} & \big\downarrow \\ \Delta^n \times I & \longrightarrow & B \end{array}$$

and we define $\gamma \cdot \sigma = \tau|_{\Delta^n \times \{1\}}$. What one should check now is that this induces a chain complex morphism which does not depend on the representative of $[\gamma] \in \pi$, and the compatibility with the concatenation of loops. Once it is done, we have a structure of $\mathbb{Z}[\pi]$-module on $H_n(F)$ for every n.

Let us now assume to have a CW-complex structure on B, and let B^p be its p-skeleton. Let then $X^p = p^{-1}(B^p)$ (despite the similarity in notation, this has no reason to be a cellular decomposition), and consider the complex $C_* = C_*(X)$ with the filtration $F_p C_* = C_*(X^p)$. By definition we have therefore $E^0_{p,q} = C_{p+q}(X^p, X^{p-1})$ and $E^1_{p,q} = H_{p+q}(X^p, X^{p-1})$.

Theorem B.2.2. There is a natural isomorphism $E^2_{p,q} \cong H_p(B; H_q(F))$, where $H_q(F)$ is endowed with the $\mathbb{Z}[\pi]$-module structure.

Proof. Omitted, see the notes above for the vector bundle case. □

Cohomology and spectral sequences

We can carry out all the operations we did in homology in cohomology too, shifting indices. The definitions will be similar, but in particular the filtration here needs to be decreasing. We define:

$$A^{p,q}_r = \{x \in F_p C^{p+q} | d \in F_{p+r} C^{p+q+1}\}$$
$$D^{p,q}_r = F_{p+1} C^{p+q} + d(F_{p-r+1} C^{p+q-1})$$
$$E^{p,q}_r = A^{p,q}_r / A^r_{p,q} \cap D^r_{p,q}$$

and we can prove the

Lemma B.2.3. A cohomological spectral sequence has the three following properties:

i) d induces a coboundary morphism on the *r-th sheet* $d^r : E^{p,q}_r \to E^{p+r,q-r+1}_r$

ii) The complexes on the *(r+1)-th* sheet are the cohomologies of the ones on the *r-th*:

$$E^{p,q}_{r+1} = \ker(d^r : E^{p,q}_r \to E^{p+r,q-r+1}_r) / \operatorname{Im}(d^r : E^{p-r,q+r-1}_r \to E^{p,q}_r)$$

iii) If the filtration is bounded, i.e. there are $p_0, p_1 \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}$ $\forall p \leq p_0$ $F_p C^n = 0$, $\forall p \geq p_1$ $F_p C^n = C_n$, then there is an $r_0 \in \mathbb{N}$ such that $\forall r \geq r_0$, $E^{p,q}_r = G_p H^{p+q}(C^*)$.

The cup-product enriches the cohomological structure here too, as described in the lemma (which we are not going to prove):

Lemma B.2.4. The cup-product $\cup$ induces, for every $n \in \mathbb{N}$, a bilinear map

$$\cup_r : E^{p,q}_r \times E^{s,t}_r \to E^{p+s,q+t}_r$$

such that:

i) $d^r(\alpha \cup_r \beta) = d\alpha \cup_r \beta + (-1)^{p+q} \alpha \cup_r d\beta$.

ii) The product $\cup_{r+1}$ is induced by $\cup_r$ taking the homology.

iii) If the filtration is bounded, $\cup_r$ stabilises to the cup-product $G_p H^q \times G_s H^t \to G_{p+s} H^{q+t}$.

Proof. We just remark that for the last property to make sense we need to prove that the cup-product is compatible with the filtration. □

To finish, one can define the cohomological Leray-Serre spectral sequence using the filtration $F^p C^*(X) := C^*(X, X^{p-1})$ (same notations as above); again, $E^{p,q}_1 = C^{p+q}(X^p, X^{p-1})$ and $E^{p,q}_2 = H^p(B; H^q(F))$.

B.3 About the existence of the Maslov class

We are going to state and prove (following [33]) a characterisation of symplectic vector bundles for which there exists a Maslov class. We start by mentioning that $H^1(Gr(Lag(\mathbb{C}^n)); \mathbb{Z})$ is generated, in fact, by the Maslov class μ. This can be seen via the discussion we made in the introductory chapter. In the following, we shall write $\Lambda(n) = Gr(Lag(\mathbb{C}^n))$. Let us now consider a symplectic vector bundle $p : E \to B$ of real rank $2n$. Let $BU(n)$ be the complex Grassmannian and $EU(n)$ the complex, unitary Stiefel manifold we defined above. Then $\eta : EU(n) \to BU(n)$ is the classifying space for $U(n)$. Since $Sp(n) \sim U(n)$, their classifying spaces coincide; we have thus a map $f : B \to BU(n)$ such that $E = f^*EU(n)$. Moreover, η factors through $\Lambda\eta : EU(n)/O(n) \to BU(n)$, and of course $EU(n)/O(n)$ can be identified with $BO(n)$. If $\Lambda E \to B$ is the lagrangian bundle on B associated to p (i.e. the fibre at each point is $\Lambda(n)$), again by what we said in the introductory chapter, $\Lambda E = f^*(EU(n)/O(n))$.

Theorem B.3.1 (Viterbo 1987)**.** There is a class $\bar{\mu} \in H^1(E; \mathbb{Z})$ representing the Maslov class on the fibres if and only if $2c_1(E) = 0$. Moreover, this class is uniquely defined up by sum with a term in $p^*H^1(B; \mathbb{Z})$.

Remark that by Axiom iii) for the Chern class, $c_1(E) \in H^2(B; \mathbb{Z})$ is in fact $f^*c_1(EU(n))$, where $c_1(EU(n)) \in H^2(BU(n))$[2].

Proof. Here we apply the Leray-Serre cohomological spectral sequence for the vector bundle $\Lambda\eta \to BU(n)$: since $\pi_1(EU(n)) = 0$ (the Stiefel manifold is weakly contractible), then the second page of the spectral sequence looks like:

$$E_2^{p,q} = H^p(BU(n); H^q(\Lambda(n))) = H^p(BU(n)) \otimes H^q(\Lambda(n))$$

Since $H^1(BU(n)) = 0$, the only elements of degree 1 lie in $H^0(BU(n)) \otimes H^1(\Lambda(n)) = \mathbb{Z}(1 \otimes \mu)$. Now, as indicated in [4], the idea is to understand at which conditions $d_2(1 \otimes \mu) = 0$ (here we apply the naturality of the spectral sequence with respect to the continuous map f, so that we are actually considering elements in the spectral sequence of ΛE). In fact, if $i : \Lambda(n) \to \Lambda E$ is the immersion of the fibre, one can prove that the image of $i^* : H^\bullet(\Lambda E; \mathbb{Z}) \to H^\bullet(\Lambda(n); \mathbb{Z})$ is given by the intersections of the kernels of all the differentials d_k in $1 \otimes H^\bullet(\Lambda(n))$. Counting degrees, $d_k = 0$ for $k \geq 3$; it is furthermore true by definition that $d^1 = 0$ on $E_2^{p,q}$. We want then $d_2(1 \otimes \mu) = 0$, so that by definition it then realises the Maslov class on the fibres via i^*. Going back to the spectral sequence of $\Lambda\eta$, the Leray-Serre sequence converges to the cohomology of $EU(n)/O(n)$, which we identified to the classifying space $BO(n)$, and in degree 1 this happens in the third page: $d_2(1 \otimes \mu) \in H^2(BO(n); R)$. It is known that $H^1(BO(n); \mathbb{Z}) = 0$ and $H^1(BO(n); \mathbb{Z}_2) \simeq \mathbb{Z}_2$; moreover, the cohomology group $H^2(BO(n), \mathbb{Z}) \simeq \mathbb{Z}_2$ (see for instance [7]) which is the reduction modulo 2 of the Chern class: this forces $d_2(1 \otimes \mu) = \pm 2c_1$, as we shall justify soon, in the spectral sequence of $EU(n)/O(n)$, and by naturality in that of ΛE too.

[2] The Chern class of the complex Grassmannian is defined to be the Chern class of the tautological bundle γ we defined above.

To prove that $d_2(1\otimes\mu) = \pm 2c_1$, it suffices to show that $E_3^{2,0} = E_2^{2,0} / \operatorname{Im} d_2 \simeq \mathbb{Z}_2$, since $E_2^{2,0} \simeq \mathbb{Z}$ is generated by the first Chern class (see [26]). To prove this, we look at the graduated $G_pH^{p+q}(BO(n);\mathbb{Z})$ for $p+q = 2$: since $H^2(BO(n);\mathbb{Z}) \simeq \mathbb{Z}_2$, if we prove that for $p \neq 2$ we have $G_pH^2(BO(n);\mathbb{Z}) = 0$, then necessarily $G_2H^2(BO(n);\mathbb{Z}) = E_3^{2,0} \simeq \mathbb{Z}_2$, which we want to show[3]. The graduated $G_pH^2(BO(n);\mathbb{Z}) = 0$ automatically when $p > 3$, since this implies $q < 0$ and that we are looking at a group in the fourth quadrant. We need to examine then $E_2^{1,1}$ and $E_2^{0,2}$. The former can be shown to be 0, using the Universal Coefficient Theorem and Hurewicz's isomorphism. Moreover, $H^2(\Lambda(n);\mathbb{Z}) = 0$ (see [1]), so that we conclude that $d_2(1\otimes\mu) = \pm 2c_1$.

For the second part, if we are given two classes $\overline{\mu}$, $\overline{\mu}'$, by constructions all the d_k are 0: applying again a remark in [4] we find the conclusion. □

[3]We can use this argument specifically because $H^2(BO(n);\mathbb{Z})$ is finite, the conclusion would not be true in general.

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