How is a quotient vector space a vector subspace?

Francesco Morabito

September 2020

When talking about vector spaces, quotients and subspaces can be identified, although in a non-canonical way, hence it might be difficult to distinguish the two notions. The aim of this document is to convince you that they are not the same.

Warning: I am not going to define all terms properly, as it would take too long and distract you from what you should try to understand. Because of this, do not focus too much on the technicalities. Ideally, you should be able to spend less than 10 minutes on this just to grasp the basic ideas, nothing more.

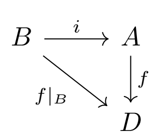
To get rid of this identification, we change "category" of objects. Let us start from scratch, working in **Set**. We shall consider sets and morphisms between sets, aka functions. Since we endow the sets with no further structure, the functions have no additional requirements: they simply map one element to another, without rules.

What is then a subset? Let A be a set, a subset can be seen as a pair (B, i) where $i : B \hookrightarrow A$ is an injection. The advantage of this approach is that way we can easily identify subsets which are isomorphic, which in the category **Set** means that they have the same cardinality.

A quotient, instead, comes with a function defined in the "dual" way. Given an equivalence relation $\mathcal{R}$ on the set A, the quotient $C := A/\mathcal{R}$ is in fact a pair (C, p), where $p : A \twoheadrightarrow C$ is the (surjective) quotient projection.

Now that the two notions are defined, we are going to see, in practice how they differ. A standard procedure to check that two objects are different in mathematics is to look at functions to and from such objects.

Let now D be a set, $f : A \to D$ be any function (to avoid traps $A, D \neq \emptyset$). How do we induce from f functions from B and C? To find the one on B, one can simply take the restriction $f|_B$; in our language it corresponds to taking $f \circ i$: the following diagram is then commutative:



With quotients we are not that lucky: what we want to achieve is to define a certain $\overline{f} : C \to D$ such that if $[a]$ denotes the equivalence class of $a \in A$,

$\overline{f}([a]) = f(a)$: this again is equivalent to requiring that the following diagram be commutative:

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & D \end{array}$$

The problem here is the good definition of f: if $a\mathcal{R}b$, we need to have $f(a) = f(b)$ to define this canonically. "Canonically" means that there are no choices involved, it is entirely natural. We say that f needs to be constant on the equivalence classes. Now, suppose it is not: we can still define a function on the quotient taking a section (a right inverse) of the projection. This amounts to choosing an element for every equivalence classes using an auxiliary function $s : C \to A$ such that $p \circ s = Id_C$, and setting $\overline{f}([a]) = f(s(a))$. Here we need to remark two points:

- such a function s always exists in **Set**: it is one of the (many) formulations of the Axiom of Choice.
- assuming that this function exists, it is injective (basic properties of functions): this is why, in **Set**, one can identify a quotient set as a subset. The identification is, as we saw, not canonical: a priori there's no particular reason to prefer one representative or the other for the same equivalence class.

Now, what could go wrong? In $\mathbf{Vect}_{\mathbb{K}}$, the category of $\mathbb{K}$-vector spaces with $\mathbb{K}$-linear maps, everything still works fine. The reason is that every vector space admits a $\mathbb{K}$-basis (again, this is equivalent to the Axiom of Choice), so that we can always define a section s (just choose a preimage for every element of the basis in the quotient vector space, and extend linearly). The fact that the operation is not canonical however remains. So, in a way, quotienting a vector space still determines a vector subspace, which is moreover a complement to the one we quotient on: if V is the ambient space, W a subspace, V/W the quotient, then

$$V \cong W \oplus V/W$$

To better see the difference between the two notions, we now look at one example where things do not go as we'd like them to, i.e. where we cannot inject a quotient into another object as a subobject. Consider **Ring** the category of (unital) rings, with ring homomorphisms (they respect addition and multiplication, map 0 to 0 and 1 to 1). As a particular example we take $\mathbb{Q}[X]$, the ring of polynomial rings in one variable over $\mathbb{Q}$. Let P, Q be two non-zero polynomials, and quotient $\mathbb{Q}[X]$ by the ideal[1] generated by the product PQ:

$$R = \mathbb{Q}[X]/(\mathbb{Q}[X]PQ)$$

[1] To have the quotient inherit a ring structure, we need to quotient by an ideal and not a subring. This does not make any difference.

The ring R is not a subring of $\mathbb{Q}[X]$. Let us assume the contrary: there is an injective morphism of rings $\varphi : R \to \mathbb{Q}[X]$. Then $\varphi(0) = 0$ and if $[a], [b] \in R$, then $\varphi([a] \cdot [b]) = \varphi([a])\varphi([b])$. But then $0 = \varphi([P] \cdot [Q]) = \varphi([P])\varphi([Q]) \neq 0$: in fact $\varphi([P]), \varphi([Q]) \neq 0$ by injectivity since $[P], [Q] \neq 0$, and $\mathbb{Q}[X]$ is an integral domain. In particular, there's no possible section of the quotient projection $p : \mathbb{Q}[X] \to R$ **which is a morphism of rings**. The important point here is that one can always choose a preimage for every equivalence class, but that it might be impossible to do it consistently with the structure we are considering on the set.